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On the Statistical Mechanics of Classical Coulomb and Dipole Gases

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A detailed, rigorous study of the statistical mechanics—screening—and critical properties, phase diagrams, etc., of classical Coulomb monopole and dipole gases in two or more dimensions is presented. The statistical mechanics of the two-dimensional XY and Villain models is reconsidered and related to the one of two-dimensional lattice Coulomb gases. At low temperatures and moderate densities those gases behave like dipole gases. The Kosterlitz–Thouless transition is analyzed in that perspective and characterized by an order parameter. Techniques useful for a proof of existence of such a transition in a two-dimensional hard-core Coulomb gas are developed and applied to the study of dipole gases.

KEY WORDS: Lattice; Coulomb and dipole gases; XY model; Kosterlitz–Thouless transition.

1. INTRODUCTION

Beginning with the work of Berezinski[1] and Kosterlitz and Thouless,[2] there have appeared numerous papers discussing the low-temperature behavior of the plane rotator and the Coulomb gas in two dimensions. There is a close connection between the rotator and the Coulomb gas made precise by Villain.[3] Roughly speaking, vortex configurations of the rotator correspond to charges in the Coulomb gas. (See Sections 2 and 3 for a review.) At high temperature the rotator always has exponential clustering and the Coulomb gas exhibits Debye screening.[4] In the case of the two-dimensional rotator, the Mermin–Wagner theorem forbids a spontane-

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ous magnetization; nevertheless there is presumed to be a temperature T_c below which correlations have only a power falloff. For the Coulomb gas at low temperature and low activity one expects that there is a phase without screening. However, there is little that is rigorously known about either of these two-dimensional models at low temperature.

In two dimensions the Coulomb potential is logarithmic, $\approx (2\pi)^{-1}\log|x|$. The long-range nature of this potential has the important consequence that it makes the Coulomb gas "locally" neutral in the following sense: if the distance between a plus charge and the nearest negative charge is r, then the contribution of such a configuration to the partition function is $\exp-(\beta/2\pi)\log r$. Moreover, there is a contribution from the entropy which is $\sim r^3$. (One factor of r^2 comes from choosing the position of the first charge and a factor of r comes from choosing the position of the second charge.) If $\beta > 8\pi$ note that

$$\int e^{-(\beta/2\pi)\log r} r^3\, dr < \infty$$

hence the total contributions of long dipoles is suppressed. (See Section 5.) Thus a natural starting point for the study of the two-dimensional Coulomb gas for large β and small fugacity is the study of dipole gases.

This article is primarily devoted to a detailed analysis of dipole gases in two and three dimensions. For dipoles of fixed length and with a hard core we show that there is no screening, provided that the fugacity is small. More precisely, we show that the charge correlations and the infinitesimal dipole correlations have a power law decay. In two dimensions we consider dipole gases in which the dipoles are allowed to assume a finite number of arbitrary lengths. The fractional charge correlation is shown to have a power law decay.

In three or more dimensions we establish the existence of an ordered phase for large fugacity, provided the dipole potential has short range. This means that the dipole correlation has a long-range order. The model we analyze for this case allows dipoles to have a continuous orientation, but the centers of the dipoles lie on a fixed lattice. If the orientations of the dipoles are constrained to be discrete, we show that, for general dipole potentials, there is a crystalline phase in two dimensions, as well. The proof of this is based on a Peierls argument of the sort used in Refs. 5 and 6 to prove the existence of a crystalline state in the two-dimensional hard core Coulomb monopole gas at low temperatures and large activities.

A key ingredient in the proof of our results is the sine-Gordon transformation. Let us consider a simple example, namely, the lattice Coulomb gas with hard core. Let $d\mu_{\beta C}$ be the Gaussian measure with covariance βC,

$$C(x, y) = (-\Delta)^{-1}(x, y)$$

and define

$$U_n(x,q) = \sum_{1 \leqslant i \leqslant j \leqslant n} C(x_i, x_j) q_i q_j$$

The transformation for the partition function in a box Λ is

$$\begin{aligned} &\int \prod_{j \in \Lambda} \left[1 + z \cos\phi(j) \right] d\mu_{\beta C}(\phi) \\ &\quad = \int \sum_{q_j = 0, \pm 1} \prod_{j \in \Lambda} z^{|q_j|} e^{iq_j\phi(j)} \, d\mu_{\beta C}(\phi) \\ &\quad = \sum_n \frac{z^n}{n!} \sum_{x_i}{}' \sum_{q_j = \pm 1} e^{-\beta U_n(x,q)} \end{aligned}$$

The sum $\sum'$ ranges over $x_j \neq x_i$.

Similarly the partition function of the lattice dipole gas with discrete orientation is given by

$$\int \prod_{j \in \mathfrak{L}_0} \left\{ 1 + z \sum_{|k-j| \approx l} \cos\left[\phi(j) - \phi(k) \right] \right\} d\mu_{\beta C} \tag{1.1}$$

Here $\mathfrak{L}_0$ is the lattice $4l\mathbb{Z}^\nu$ and l denotes the length of the dipoles. We use this representation together with Mermin–Wagner type[7,8] methods to establish upper bounds on fractional charge correlations and lower bounds on ϕ correlations in momentum space. However, if z is not small, notice that the resulting measure in ϕ space is not positive and our estimates break down when applied to (1.1). For this reason, in two dimensions it is helpful to go to a modified representation in which (1.1) is replaced by

$$\int \prod_j \left\{ 1 + \bar{z} \sum_k \cos\left[\bar{\delta}\phi_k(j) \right] \right\} d\mu_{\beta C} \tag{1.2}$$

with

$$|\bar{z}| \leqslant \mathrm{const}|z| e^{-\beta(\log l)/2\pi}$$

and $\bar{\delta}\phi$ is defined in Section 5. The identity between (1.1) and (1.2) is obtained by using a mixture of the ϕ and charge q representations. Since ϕ and q are dual variables, our analysis can be thought of as a phase space analysis in function space.

Let us consider the fractional charge correlation,

$$\langle e^{i\alpha[\phi(0) - \phi(x)]} \rangle_\phi \tag{1.3}$$

in some more detail. We shall show in two dimensions that, for both the Coulomb and dipole expectations, (1.3) is bounded below by $|x|^{-\alpha^2\beta/2\pi}$ for all activities $z \geqslant 0$. This bound is a consequence of Jensen's inequality. Of course for small β we know that the truncated correlation in the Coulomb gas clusters exponentially [4]. This means that the truncation must be nontrivial, i.e.,

$$\langle e^{i\alpha\phi(0)} \rangle \neq 0$$

Thus $\langle e^{i\alpha\phi(0)} \rangle$ should be regarded as an *order parameter* for the Coulomb or sine-Gordon models, and we shall see that the role of boundary conditions is crucial. Our aim is to show that if β is large $\langle e^{i\alpha[\phi(0)-\phi(x)]} \rangle$ goes to zero for large x. Thus far we have only succeeded in proving this for dipole systems, but we believe that our technique will enable us to eventually extend the result to the Coulomb case. The technique is to expand the Coulomb gas in terms of gases of neutral multipoles by means of some sort of "block spin" transformations. It is important to note that by the above arguments we have reduced the proof of existence of such a phase transition to proving an *upper bound* on a correlation function, as opposed to the more difficult proofs of lower bounds. Moreover, the fractional charge correlation is extremely useful in the analysis of the Coulomb gas in two dimensions, because it really looks like a charge–charge correlation in a sea of dipoles. An integral charge in a Coulomb gas would tend to pair with an opposite charge and thus the correlation would behave like a dipole–dipole correlation in a sea of dipoles, which requires a much more subtle analysis.

We conclude this introduction with a short summary of the different sections of this paper: our main new results are in Sections 4, 5, and 7, but see also Section 6.

In Section 2 we review the sine-Gordon (or Siegert) transformation, i.e., the passage from the q to the ϕ representation, in a form convenient for our purposes. We also recall integration by parts on function space, in the ϕ representation, which is important for later sections. Another piece of abstract formalism, reflection positivity (in the ϕ and q representations), is reviewed in Appendix A. It is applied to establish an analog of superstability estimates (the chessboard estimates) for classical Coulomb systems and infrared bounds used to prove the existence of phase transitions with order parameter (see Sections 4 and 7).

In Section 3 we review the main rigorous results on the two-dimensional rotator and Villain models (Theorems 3.1–3.5) and describe the Kosterlitz–Thouless transition (Conjectures 3.2 and 3.2^v). For comparison, some rigorous, partly new results on general N-vector models, $N > 2$, are quoted [(3.18)–(3.19)]. The duality (Fourier) transformation of the rotator and Villain model is recalled (Theorem 3.6), and the isomorphism between Villain model and Coulomb gas is described. That Coulomb gas is shown to be a limiting ensemble of a family of Coulomb gas ensembles labeled by an activity, z, as $z \to \infty$ (Section 3.3, Theorem 3.8).

In Section 4, classical Coulomb gases in different ensembles are studied in some detail. In Section 4.1, the screening properties, the inverse correlation length (mass), convexity, and decay properties of the charge two-point correlation and the phase diagram (existence of ordered states) of those Coulomb gases are discussed. The main results are summarized in Theorems 4.1–4.5. In Section 4.2, we specialize to the two-dimensional Coulomb gas. We give several different characterizations of the Kosterlitz–Thouless transition and discuss its relation to the roughening transition. This complements the discussion of that transition for the rotator and Villain model in Sections 3.1 and 3.2.

In Section 5 we study the behavior of the fractional charge correlation and the expectation value of the disorder parameter in several different two-dimensional dipole gases, in particular in a gas of dipoles of various lengths that mimicks the two-dimensional hard core Coulomb gas at low density (z small) and low temperature. We prove upper and lower bounds with power law decay. A method for renormalizing the dipole activities, based on estimating dipole self-energies and replacing dipoles by neutral multipoles of larger size, is developed, and its workings demonstrated. That method combined with complex translations of the ϕ variables in the functional integral expressing the fractional charge correlation in the ϕ representation yields our main decay estimates on that correlation. In Appendix B an alternate (purely electrostatic) method for renormalizing the activities of neutral dipoles is sketched. The emphasis in Section 5 is placed on concepts and analytical tools rather than on optimal results. We believe that the techniques of Section 5 will eventually permit us to prove convergence of an expansion of the two-dimensional Coulomb gas in terms of neutral multipole configurations, at low density and low temperature, designed to imply the existence of the Kosterlitz–Thouless transition. But the required combinatorial and refined electrostatic estimates are still missing.

In Section 6 we establish absence of screening in general dipole gases, in the unordered phase (Theorem 6.1, Applications 1, 2). Our main tool is a generalized version of the Mermim or Goldstone theorem (Theorem 6.3). The basic reason why the "Goldstone theorem" applies and there is no screening lies in the fact that dipole gases have a spontaneously broken, continuous symmetry, $\phi \to \phi + \text{const}$, manifest in the ϕ representation. We also use our version of Mermin's theorem to prove mean field lower bounds on the magnetization in continuous spin lattice systems (Section 6, Application 4).

In Section 7 we study a general class of lattice dipole potentials, estimate Madelung constants, i.e., energies of periodic dipole configurations, analyze the ground-state configurations, and prove infrared bounds on the truncated dipole–dipole correlation in momentum space. All this serves to establish the existence of phase transitions with order parameter and of ordered states (oppositely oriented, infinite chains of aligned dipoles) at high density and low temperature, for various classes of hard core dipole gases. Depending on dimension and dipole ensemble we use the infrared bound method ($\nu \geqslant 3$, short-range dipole potentials, orientation of dipoles continuous) or the Peierls chessboard method ($\nu \geqslant 2$, long-range dipole potentials, dipole orientation discrete). The material in Section 7 is rather intricate, and we recommend that, in a first reading, only the main definitions and results be studied.

2. THE SINE-GORDON OR SIEGERT TRANSFORMATION[9,10]: FOURIER TRANSFORMATION IN THE CHARGE VARIABLES

In this section we review a well-known formulation of the statistical mechanics of classical gases of particles interacting through two-body potentials of positive type in terms of Gaussian integrals: via functional Fourier transformation the charges of classical particles are traded for conjugate variables. This formalism has proven to be very useful; see, e.g., Refs. 4, 10, and 11. It is a basic tool of the present paper, as well (permitting localization in "phase space"). We then recall correlation inequalities of Ref. 11, and integration by parts on function space,[12] and we give a preview of applications. In an appendix to Section 2 (Appendix A) we review reflection positivity.[13,6]

2.1. Functional Integrals and Statistical Mechanics, Inequalities

Let $\mathcal{C}$ be the configuration space of one classical extended or point particle. In this paper $\mathcal{C}$ will usually be a lattice, $\mathfrak{L}$, in particular $\mathcal{C} = \mathbb{Z}^\nu$, but for later purposes (see, e.g., Sections 6 and 7) we admit the possibility that $\mathcal{C} = \mathbb{R}^\nu$. Points in $\mathcal{C}$ are denoted $x, y, \ldots,$ and dx is the counting measure on $\mathfrak{L}$ if $\mathcal{C} = \mathfrak{L}$, or the Lebesgue measure on $\mathbb{R}^\nu$ if $\mathcal{C} = \mathbb{R}^\nu$.

Let $\bar{\mathfrak{L}} \subseteq \mathcal{C}$ be some lattice and let $\{\Delta_x\}_{x \in \bar{\mathfrak{L}}}$ be a cover of $\mathcal{C}$ by disjoint hypercubes (squares for $\nu = 2$, cubes for $\nu = 3, \ldots$) with sides parallel to the axes of $\bar{\mathfrak{L}}$ and centered at the sites of $\bar{\mathfrak{L}}$. The possible positions of one classical particle are identified with the sites of $\bar{\mathfrak{L}}$.

Let Q_0 be some measurable space of distributions, ρ_0, with support inside Δ_0. Let $d\lambda$ be some measure on Q_0. Given a distribution $\rho_0 \in Q_0$, we define ρ_x by

$$\rho_x(y) = \rho_0(y - x) \tag{2.1}$$

Clearly $\operatorname{supp} \rho_x \subseteq \Delta_x$. We define Q_x to be the space of all distributions ρ_x obeying (2.1) for some $\rho_0 \in Q_0$, and

$$d\lambda_x(\rho_x) \equiv d\lambda(\rho_x) = d\lambda(\rho_0) \tag{2.2}$$

A distribution $\rho_x \in Q_x$ is interpreted as the *charge distribution* of a classical particle located at x. The measure $d\lambda_x$ assigns an *a priori* weight to each charge distribution.

Next, let $C(x, y)$ be the kernel of a positive (semi-)definite quadratic form, C, on $L^2(\mathcal{C}, dx)$. We assume throughout this paper that

$$C(x, y) \quad \text{is real-valued and continuous in } x \text{ and } y. \tag{2.3}$$

Let $Q(C) \subseteq L^2(\mathcal{C}, dx)$ denote the quadratic form domain of C, and $\mathcal{H}_{\beta C}$ the closure of $Q(C)$ in the scalar product

$$\langle f, g\rangle_{\beta C} = \beta(f, Cg)_{L^2(\mathcal{C},dx)} \tag{2.4}$$

f, g in $Q(C)$.

Subsequently, β is interpreted as the *inverse temperature*, and $C(x, y)$ is the potential between two point particles of charge 1, located at x or y.

Let $\phi = \phi(x)$ be the Gaussian process with mean 0 and covariance βC indexed by $\mathcal{H}_{\beta C}$. The distribution of ϕ is the Gaussian measure

$$d\mu(\phi) \equiv d\mu_{\beta C}(\phi)$$

with mean 0 and covariance βC.

The expectation in $d\mu$ is denoted $\langle \cdot \rangle_{\beta C}$. By definition

$$\langle\phi(f)\rangle_{\beta C} = 0, \quad \langle\phi(f)\phi(g)\rangle_{\beta C} = \langle f, g\rangle_{\beta C}, \tag{2.5}$$

where $\phi(f) = \int_{\mathcal{C}}\phi(x)f(x)dx$, and f is a test function (e.g., Schwartz space function) on $\mathcal{C}$. By power series expansion one finds, using (2.5),

$$\langle e^{i\phi(f)}\rangle_{\beta C} = \exp\left[-\tfrac{1}{2}\langle f, f\rangle_{\beta C}\right] \tag{2.6}$$

Wick ordering is defined by

$$:e^{i\phi(f)}:_{\beta C} = e^{i\phi(f)}\langle e^{i\phi(f)}\rangle_{\beta C}^{-1} \tag{2.7}$$

We now suppose that $\rho_x \in \mathcal{H}_{\beta C}$, for all $\rho_x \in Q_x$ and all $x \in \bar{\mathcal{L}}$. Then Eqs. (2.6) and (2.7) make sense for $f = \sum_{x\in\bar{\mathcal{L}}} \rho_x, \rho_x \in Q_x, \rho_x = 0$ except for finitely many x. From those equations follows the lemma below.

Lemma 2.1

$$\left\langle \prod_{j=1}^{n} e^{i\phi(\rho_{x_j})} \right\rangle_{\beta C} = \exp\left[-(\beta/2) \sum_{i,j=1}^{n} (\rho_{x_i}, C\rho_{x_j})\right] \tag{2.8}$$

$$\left\langle \prod_{j=1}^{n} :e^{i\phi(\rho_{x_j})}:_{\beta C} \right\rangle_{\beta C} = \exp\left[-\beta \sum_{1 \leqslant i<j \leqslant n} (\rho_{x_i}, C\rho_{x_j})\right] \tag{2.9}$$

We notice that the right-hand side of (2.8) is the Gibbs factor of n classical particles located at points $x_1, \ldots, x_n$ in $\tilde{\mathfrak{L}}$ with charge distributions $\rho_{x_1}, \ldots, \rho_{x_n}$, interacting through two-body forces with potential $C(x, y)$. In (2.8) the self-energies of these particles are included; in (2.9) they are omitted.

Next, we define functions

$$F(\phi_x) = \int_{Q_x} d\lambda(\rho_x) e^{i\phi(\rho_x)}, \qquad x \in \tilde{\mathfrak{L}} \tag{2.10}$$

of the Gaussian process ϕ.

Note that $F(\phi_x)$ is localized in the hypercube Δ_x [i.e., if $\phi(\cdot)$ and $\phi'(\cdot)$ are two samples in the support of $d\mu_{\beta C}$ with $\phi(y) = \phi'(y)$, for $y \in \Delta_x$, then $F(\phi_x) = F(\phi'_x)$]. Moreover $F(\phi_x)$ is obtained from $F(\phi) = \int_{Q_0} d\lambda(\rho_0) e^{i\phi(\rho_0)}$ by the substitution $\phi(y) \to \phi_x(y) = \phi(y - x)$. This follows from (2.1) and (2.2). We set

$$F_\Lambda(\phi) = \prod_{x \in \Lambda} F(\phi_x),$$

where Λ is a finite subset of $\tilde{\mathfrak{L}}$, and introduce the measure

$$\langle F_\Lambda \rangle_{\beta C}^{-1} F_\Lambda(\phi) d\mu_{\beta C}(\phi). \tag{2.11}$$

Expectation in this measure is denoted $\langle\ \rangle(\beta; F)$.

Lemma 2.2. Let Λ be a bounded region in $\tilde{\mathfrak{L}}$. Then

$$\Xi_\Lambda(\beta; F) \equiv \langle F_\Lambda \rangle_{\beta C} = \int \prod_{x \in \Lambda} d\lambda(\rho_x) \exp\left[-(\beta/2) \sum_{y, y' \in \Lambda} (\rho_y, C\rho_{y'}) \right] \tag{2.12}$$

$$\begin{aligned}\left\langle \prod_{i=1}^{n} e^{i\phi(\tilde{\rho}_{x_i})} \right\rangle_\Lambda (\beta; F) &= \Xi_\Lambda(\beta; F)^{-1} \int \prod_{x \in \Lambda} d\lambda(\rho_x) \\ &\times \exp\left[-(\beta/2) \sum_{y, y' \in \Lambda} (\rho_y, C\rho_{y'}) \right] \exp\left[-\beta \sum_{\substack{i=1 \\ y \in \Lambda}}^{n} (\rho_y, C\tilde{\rho}_{x_i}) \right] \\ &\times \exp\left[-(\beta/2) \sum_{i,j=1}^{n} (\tilde{\rho}_{x_i}, C\tilde{\rho}_{x_j}) \right]\end{aligned} \tag{2.13}$$

Clearly, $\Xi_\Lambda(\beta; F)$ is the *partition function*, and $\langle \prod_{i=1}^{n} e^{i\phi(\tilde{\rho}_{x_i})} \rangle_\Lambda(\beta; F)$ the correlation functions of a system of classical particles in the region Λ with charge distributions $\tilde{\rho}_x \in Q_x$ and interaction potential $C(x, y)$, at inverse temperature β. The expectation $\langle \cdot \rangle_\Lambda(\beta; F)$ is the equilibrium expectation. Lemma 2.2 is a direct consequence of Lemma 2.1; see also Refs. 9 and 10.

Next, suppose that $A(\rho)$ is a function on $\prod_{x\in\Lambda} Q_x$. We define

$$\langle A\rangle_\Lambda(\beta;F)=\Xi_\Lambda(\beta;F)^{-1}\int\prod_{x\in\Lambda}d\lambda(\rho_x)$$

$$\times\exp\Big[-(\beta/2)\sum_{y,y'\in\Lambda}(\rho_y,C\rho_{y'})\Big]A(\rho)$$

$$=\langle F_\Lambda\rangle_C^{-1}\Big\langle\int\Big[\prod_{x\in\Lambda}d\lambda(\rho_x)e^{i\phi(\rho_x)}\Big]A(\rho)\Big\rangle_{\beta C} \tag{2.14}$$

and (2.14) follows from Lemma 2.1 and Fubini's theorem (provided $d\lambda$ is a finite measure). The expectation

$$\langle\rho_x\rho_y\rangle_\Lambda(\beta;F)\qquad[\text{i.e., } A(\rho)=\rho_x\rho_y]$$

is called charge two-point correlation.

Next, we consider the case where

$$d\lambda(\rho_x)=d\lambda(-\rho_x) \tag{2.15}$$

(2.15) is a neutrality condition expressing charge conjugation invariance.

Lemma 2.3. Assume that $d\lambda$ satisfy the neutrality condition (2.15). Then

$$\Big\langle\prod_{i=1}^n e^{i\phi(\tilde\rho_{x_i})}\Big\rangle_\Lambda(\beta;F)\geqslant\Big\langle\prod_{i=1}^n e^{i\phi(\tilde\rho_{x_i})}\Big\rangle_{\beta C}$$

for arbitrary Λ.

Proof. By Lemma 2.2, (2.14), and Jensen's inequality,

$$\Big\langle\prod_{i=1}^n e^{i\phi(\tilde\rho_{x_i})}\Big\rangle_\Lambda(\beta;F)$$

$$\geqslant\exp\Big[-\beta\Big\langle\sum_{\substack{i=1\\ y\in\Lambda}}^n(\rho_y,C\tilde\rho_{x_i})\Big\rangle_\Lambda(\beta;F)\Big]$$

$$\times\exp\Big[-(\beta/2)\sum_{i,j=1}^n(\tilde\rho_{x_i},C\tilde\rho_{x_j})\Big]$$

By (2.14) and (2.15),

$$\Big\langle\sum_{y\in\Lambda}(\rho_y,C\tilde\rho_{x_i})\Big\rangle_\lambda(\beta;F)=0\qquad\text{for all } i=1,\ldots,n$$

Finally

$$\exp\left[-(\beta/2)\sum_{i,j=1}^{n}(\tilde{\rho}_{x_i}, C\tilde{\rho}_{x_j})\right] = \left\langle \prod_{i=1}^{n} e^{i\phi(\tilde{\rho}_{x_j})}\right\rangle_{\beta C} \qquad \blacksquare$$

Next, we consider two special ensembles. We suppose that $d\lambda$ is a probability measure, and z is a positive number.

(*Gnhc—General No-Hard-Core Ensemble*) We set

$$F(\phi_x) = \exp\left[z\int_{Q_x} d\lambda(\rho_x)\cos\phi(\rho_x)\right] \qquad \text{(I)}$$

$$\text{(2.16)}$$

(*Ghc—General Hard Core Ensemble*)

$$F(\phi_x) = 1 + z\int_{Q_x} d\lambda(\rho_x)\cos\phi(\rho_x) \qquad \text{(II)}$$

Since $\cos\phi(\rho) = \cos\phi(-\rho)$, these ensembles are automatically charge conjugation invariant. The interest in the (Gnhc) ensemble is motivated by the following theorem.

Theorem 2.4. In the (Gnhc) ensemble (2.16), (I) $\langle\prod_{i=1}^{n} e^{i\phi(\tilde{\rho}_{x_i})}\rangle_\Lambda(\beta, z)$ is monotone increasing in z and Λ and decreasing in βC; $\langle|\phi(f)|^2\rangle_\Lambda(\beta, z)$ is decreasing in z and Λ and increasing in βC.

Remark. As explained in Ref. 11, Theorem 2.4 serves to construct the thermodynamic limit, $\Lambda \to \bar{\mathfrak{L}}$, and to derive monotonicity properties of critical temperatures, susceptibilities, etc. in z and C.

It is shown in Ref. 11 that under suitable assumptions on Q_x, $d\lambda$, and C, the (Gnhc) ensemble has a continuum limit, $\bar{\mathfrak{L}} \to \mathbb{R}^\nu$.

For a somewhat different treatment of the sine-Gordon transformation and complete proofs see Ref. 11.

2.2. Monopole and Dipole Gases

In this section we specialize to monopole and dipole gases.

(M) For *monopoles*,

$$Q_x = \{\delta_x(y) : q \in \mathbb{R}\}$$

with

$$\delta_x(y) = \begin{cases} \delta_{xy} & \text{if } \mathcal{C} = \mathfrak{L} \\ \delta(x-y) & \text{if } \mathcal{C} = \mathbb{R}^\nu \end{cases} \qquad (2.17)$$

The measure $d\lambda$ on Q is induced by a measure on the real line which we also denote by $d\lambda$. A typical example for $d\lambda$ is

$$d\lambda(q) = \left[\sum_{m \in \mathbb{Z}} c_m \delta(q - m) \right] dq \tag{2.18}$$

where $\{c_m\}$ is a bounded sequence of nonnegative numbers.

We distinguish three different ensembles:

(Mnhc), the grand canonical ensemble for monopoles *without hard cores* obtained from Eqs. (2.11) and (2.16), (I) by setting

$$F(\phi_x) = \exp[z \cos \phi(x)] \tag{2.19}$$

i.e., $d\lambda(\rho_x)$ assigns weight 0 to all distributions $\rho_x \in Q_x$, except $\rho_x = \delta_x$. The equilibrium expectation $\langle \cdot \rangle_\Lambda(\beta; F)$ is now denoted $\langle \cdot \rangle_\Lambda(\beta, z)$. By Theorem 2.4, $\langle \cdot \rangle_\Lambda(\beta, z)$ has a thermodynamic limit, $\langle \cdot \rangle(\beta, z)$. The parameter z is interpreted as the *activity* of a monopole. The continuum limit of the (Mnhc) ensemble is discussed in Refs. 10 and 11.

(Mhc), the grand canonical ensemble for monopoles *with hard cores* obtained from (2.11) and (2.16), (II) by setting

$$F(\phi_x) = 1 + z \cos \phi(x) \tag{2.20}$$

The equilibrium expectation is denoted by $\langle \cdot \rangle^{\mathrm{hc}}(\beta, z)$. Each site $x \in \tilde{\mathfrak{L}}$ can be occupied by at most one monopole of charge ± 1 and activity z.

(Mg), a general equilibrium ensemble for monopoles obtained from (2.11) by setting

$$F(\phi_x) = \int_{\mathbb{R}} d\lambda(q) e^{iq\phi(x)} \tag{2.21}$$

The (Mhc) and (Mg) ensembles generally do not have a well-defined continuum limit, and their phase diagrams are more complicated than the one of the (Mnhc) ensemble. The phase diagram of the (Mhc) ensemble has the following features: (i) For small β, z not too large, and C the Coulomb potential, the equilibrium expectation is unique, and there is exponential Debye screening.[4] (ii) For some class of reflection positive (RP) translation-invariant potentials C, $z = O(e^{\beta/2C(0)})$ and β large, one encounters the formation of a ladder crystal as shown in Ref. 6; see also Section 4. (iii) It is expected that in two dimensions, with C the Coulomb potential, there is a dilute, translation-invariant low-temperature phase where screening breaks down (formation of dipoles), for β large and $z = O(1)$. This phase is characterized in Sections 3 and 4. We hope to prove its existence predicted in Refs. 1 and 2 in a future paper.

The phase diagram of the (Mnhc) ensemble is simpler in so far as (ii) is absent. The (Mg) ensemble interpolates between (Mhc) and (Mnhc). In the study of (iii) dipole gases play an important role.

We define analogous ensembles for the dipole gases.

(D) For *dipoles*,

$$Q_x = \begin{cases} q(\delta_{x+r} - \delta_x), & r \in \Delta_0 \subset \mathcal{C}, \quad q \in \mathbb{R} \\ \left[(q \cdot \partial)\delta\right]_x, & q \in \mathbb{R}^\nu \qquad \text{if } \mathcal{C} = \mathbb{R}^\nu \end{cases} \tag{2.22}$$

where in the second case

$$(q \cdot \partial) = \sum_{\alpha=1}^{\nu} q^\alpha(\partial/\partial x^\alpha) \tag{2.23}$$

and it is assumed then that $(\Delta_x C)(x, y)$ is continuous in x and y.

The measure $d\lambda$ on Q_x is induced by, respectively, a measure $d\lambda(q, r)$ on $\mathbb{R} \times \Delta_0$, a measure $d\lambda(q)$ on $\mathbb{R}^\nu$, e.g., $d\lambda(q) = \delta(|q|^2 - 1)d^\nu q$. We set

$$(\delta_r\phi)(x) = \phi(x + r) - \phi(x) \tag{2.24}$$

The (Dnhc) and (Dhc) grand canonical ensembles are then defined as in (2.16), (I), (II), in perfect analogy to the (Mnhc) and (Mhc) ensembles. An example for a (Dg) ensemble is

$$F(\phi_x) = \int_{\mathbb{R}^\nu} d\lambda(q) e^{i[(q\cdot\partial)\phi](x)}, \qquad x \in \bar{\mathcal{L}} \tag{2.25}$$

$$\langle\,\cdot\,\rangle_\Lambda(\beta; F) = \langle F_\Lambda\rangle_{\beta C}^{-1}\langle - F_\Lambda\rangle_{\beta C} \qquad \text{with } F_\Lambda = \prod_{x\in\Lambda\subset\bar{\mathcal{L}}} F(\phi_x) \tag{2.26}$$

see (2.11) and Lemma 2.2.

The phase diagrams of dipole gases are somewhat simpler than the one of monopole gases: if C is the Coulomb potential the dipole gases have *no* phase with Debye screening, a new result which we prove in Sections 5 and 6 by using the ϕ representation (sine-Gordon transformation) to exhibit a spontaneously broken, continuous symmetry: $\phi \to \phi + \text{const}$. The (Dhc) and (Dg) ensembles have generally an interesting low-temperature phase: for large density, $z = O(e^{(\beta/2)C(0)})$, and $\beta << 1$, and ordered (crystalline) equilibrium state appears, for general distributions $d\lambda(q)$ on $\mathbb{R}^\nu$, including *rotation-invariant* ones ($\nu \geqslant 3$) (when $\nu = 2, d\lambda$ must be assumed to be discrete). This result is proven in Section 7.

2.3. Integration by Parts Formula

In this section we recall a standard integration by parts formula.[13] Let ϕ be the Gaussian process determined by

$$\langle\phi(x)\rangle_{\beta C} = 0, \qquad \langle\phi(x)\phi(y)\rangle_{\beta C} = \beta C(x, y)$$

Let F be some measurable function of $\phi(\cdot)$. Then

$$\langle\phi(x)F\rangle_{\beta C} = \beta\int_{\mathcal{C}} dy\, C(x, y)\left\langle \frac{\partial F}{\partial\phi(y)} \right\rangle_{\beta C} \tag{2.27}$$

and consequently

$$\langle\phi(x)\phi(y)F\rangle_{\beta C} = \beta C(x,y)\langle F\rangle_{\beta C} + \beta^2\int\int dz\,dz'\,C(x,z)C(y,z')\left\langle \frac{\partial^2 F}{\partial\phi(z)\partial\phi(z')}\right\rangle_{\beta C} \tag{2.28}$$

The *proof* of (2.27) and (2.28) is standard: one approximates the Gaussian functional integral by a finite-dimensional Gaussian integral for which (2.27) and (2.28) are the standard integration by parts formulas. For details see, e.g., Ref. 12.

Next, let $F = F_\Lambda$ be the multiplicative functional (2.21) defining the monopole ensemble (Mg). Then Eq. (2.28) gives

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle_\Lambda(\beta;F) &= \langle F_\Lambda\rangle_{\beta C}^{-1}\langle\phi(x)\phi(y)F_\Lambda\rangle_{\beta C}\\ &= \beta C(x,y) - \beta^2\int\int C(x,z)C(y,z') \qquad (2.29)\\ &\quad \left[\int\prod_{u\in\Lambda} d\lambda(q_u)q_z q_{z'}\left\langle\prod_{v\in\Lambda} e^{iq_v\phi(v)}\right\rangle_{\beta C}\right]dz\,dz'\\ &= C(x,y) - \beta^2\int\int C(x,z)C(y,z')\langle q_z q_{z'}\rangle_\Lambda(\beta;F)\,dz\,dz'\end{aligned}$$

and we have used (2.21) and (2.14).

Here $\langle q_z q_{z'}\rangle_\Lambda(\beta;F)$ is the usual charge–charge correlation (two-point) function.

By smearing out both sides of (2.29) we get

$$\langle|\phi(f)|^2\rangle_\Lambda(\beta;F) = \beta(f,Cf) - \beta^2\langle|(C*q)(f)|^2\rangle_\Lambda(\beta;F)$$

in particular,

$$\langle|\phi(f)|^2\rangle_\Lambda(\beta;F) < \beta(f,Cf)$$

Moreover, we conclude from (2.29) that

$$\langle|(C*q)(f)|^2\rangle_\Lambda(\beta;F) \leqslant \beta^{-1}(f,Cf) \tag{2.30}$$

provided $\langle|\phi(f)|^2\rangle_\Lambda(\beta;F) \geqslant 0$ for which it suffices that $F(\phi_x) \geqslant 0$ (e.g., $0 \leqslant z \leqslant 1$ in the (Mhc) ensemble).

If C and $\langle\cdot\rangle_\Lambda(\beta;F)$ are translation invariant [e.g., $\langle\cdot\rangle_{\Lambda=\mathcal{E}}(\beta;F)$ a translation-invariant thermodynamic limit, or periodic boundary conditions at $\partial\Lambda$] then we obtain from (2.30) by Fourier transformation

$$\langle\hat{q}(k)\hat{q}(-k)\rangle_\Lambda(\beta,F) \leqslant [\beta\hat{C}(k)]^{-1} \tag{2.31}$$

We now specialize to the (Mnhc) and (Mhc) ensembles. Let

$$K_{\beta,z}(u)=\begin{cases}\langle\cos\phi(u)\rangle_\Lambda(\beta,z) & \text{for (Mnhc)}\\ \left\langle\dfrac{\cos\phi(u)}{1+z\cos\phi(u)}\right\rangle(\beta,z) & \text{for (Mhc)}\end{cases}$$

and

$$S(x)=\begin{cases}\sin\phi(x) & \text{for (Mnhc)}\\ \dfrac{\sin\phi(x)}{1+z\cos\phi(x)}\end{cases}$$

Then

$$\begin{aligned}\langle\phi(x)\phi(y)\rangle(\beta,z)&=\beta C(x,y)-\beta^2 z\int du\,C(x,u)C(y,u)K_{\beta,z}(u)\\ &\quad+\beta^2z^2\int\int du\,du'\,C(x,u)C(y,u')\\ &\qquad\times\langle S(u)S(u')\rangle_\Lambda(\beta,z)\\ &=\beta C(x,y)-\beta^2\langle(C*q)(x)(C*q)(y)\rangle_\Lambda(\beta,z)\end{aligned}\tag{2.32}$$

i.e., in the translation-invariant case

$$\langle|\hat S(k)|^2\rangle_\Lambda(\beta,z)=(1/z)K_{\beta,z}(0)-(1/z^2)\langle|\hat q(k)|^2\rangle_\Lambda(\beta,z)\tag{2.33}$$

and

$$\langle|\phi(f)|^2\rangle_\Lambda(\beta,z)\geqslant\beta(f,Cf)-\beta^2 zK_{\beta,z}(0)(f,C^2f)\tag{2.34}$$

Equations (2.32) and (2.33) are useful in the discussion of Debye screening (sum rules and upper bound on physical mass) and of absence of long- and short-range order in $\langle q_xq_y\rangle(\beta,z)$. See Section 4.1.

Of course, the same identities can be applied to dipole gases: for the (Dg) ensemble on a lattice $\mathfrak{L}$ one finds in the translation-invariant case

$$\begin{aligned}\langle(\partial\phi)\cdot(x)(\partial\phi)(y)\rangle(\beta;F)&=\beta(-\Delta C)(x-y)\\ &\quad-\beta^2\sum_{z;i,j}\langle q_i(x)q_j(z)\rangle(\beta;F)\,W^{ij}(z-y)\end{aligned}\tag{2.35}$$

with $W^{ij}(x)=(\partial^i\partial^j\Delta C*C)(x)$ the dipole potential. The second term on the right-hand side of (2.35) is positive definite. When ∂^i,∂ are finite difference derivatives, and C is the Green's function of the finite difference Laplacean this yields

$$\langle(\partial\phi)(x)(\partial\phi)(y)\rangle(\beta,F)=\beta\delta_{xy}-\beta^2\sum_{z;i,j}\langle q_i(x)q_j(z)\rangle(\beta;F)\,W^{ij}(z-y)$$

where $W^{ij}(z-y)$ is the usual lattice dipole potential.

As in the (M) ensembles one may finally apply integration by parts on function space in order to prove an identity analogous to (2.32) which yields an inequality analogous to (2.34), namely,

$$\langle|\phi(f)|^2\rangle \geqslant \beta(f, Cf) - \beta^2 z\int_{\mathbb{R}\times\Delta_0} d\lambda(q,r)q^2K(r;\ \beta,z)\big(f,(\delta_r C)^2 f\big) \quad (2.36)$$

where, in the translation-invariant case,

$$K(r;\ \beta,z) = \begin{cases} \langle\cos(\delta_r\phi)(0)\rangle(\beta,z) & \text{for (Dnhc)} \\ \left\langle \dfrac{\cos(\delta_r\phi)(0)}{1+z\cos(\delta_r\phi)(0)}\right\rangle(\beta,z) & \text{for (Dhc)} \end{cases}$$

Inequality (2.36) together with obvious bounds on $K(r;\ \beta,z)$ provide an easy proof of the absence of screening in dipole gases for $\beta < O(1/z)$: by a chessboard estimate[14] one can show that

$$|K(r;\ \beta,z)| \leqslant c_l e^{-c_2\beta\log r}$$

for some positive constants c_1, c_2, and $0 \leqslant z \leqslant 1$ in the (Dhc) case. This combined with (2.36) yields *absence of screening* for $\beta < \text{const}\,1/z$ and for β sufficiently large, depending on z. In Sections 5 and 6 we devise much stronger methods which prove absence of screening for *all* β and z and yield more explicit information.

Finally, we wish to draw attention to the following upper bound on the dipole–dipole correlation which follows from a somewhat different form of the sine-Gordon transformation, used, e.g., in Ref. 11, by means of integration by parts: suppose that $F(\phi) \geqslant 0$ [i.e., $z < 1$ in the (Dhc) ensemble]. Then

$$\langle q(\bar{f})q(f)\rangle_\Lambda(\beta; F) \leqslant \beta^{-1}(f, W^{-1}f) \quad (2.37)$$

where

$$q(f) = \sum_{j\in\Lambda}\sum_{\alpha=1}^{\nu} q_j^\alpha f^\alpha(j)$$

and W is the dipole–dipole potential.

For other application of integration by parts see Section 4. In Appendix A we review the concept of reflection positivity which plays a basic role in Sections 3, 4, and 7. That appendix may be skipped in a first reading.

3. CONNECTIONS BETWEEN THE CLASSICAL ROTATOR (XY) MODEL, THE VILLAIN MODEL, AND COULOMB GASES

3.1. The Classical XY Model: A Review

The rotator or classical XY model is the following classical lattice spin system:

We choose $\mathfrak{L} = \mathbb{Z}^\nu$, $\nu = 2, 3(4, \ldots)$. To each site $x \in \mathfrak{L}$ we assign a two-component unit vector, S_x, interpreted as a "classical spin."

$$\mathbf{S}_x = (S_x^1, S_x^2) = (\cos\theta_x, \sin\theta_x), \qquad \theta_x \in [0, 2\pi] \tag{3.1}$$

Clearly

$$\mathbf{S}_x \cdot \mathbf{S}_y = \cos(\theta_x - \theta_y) = \mathrm{Re}\left[e^{i(\theta_x - \theta_y)}\right] \tag{3.2}$$

The *a priori* distribution of $\mathbf{S}_x$ is the uniform measure on the circle, i.e., $d\theta_x/2\pi$. The classical Hamilton function of the system constrained to a bounded region $\Lambda \subset \mathbb{L}$ is defined by

$$\begin{aligned} H_\Lambda \equiv H(\theta_\Lambda) &= -\sum_{xy \subset \Lambda} \mathbf{S}_x \cdot \mathbf{S}_y - h \sum_{x \in \Lambda} S_x^1 \\ &= -\sum_{xy \subset \Lambda} \cos(\theta_x - \theta_y) - h \sum_{x \in \Lambda} \cos\theta_x \end{aligned} \tag{3.3}$$

where xy are nearest neighbors, and h is an external magnetic field.

The equilibrium state at inverse temperature β is given by the measure

$$Z_\Lambda(\beta, h)^{-1} e^{-\beta H(\theta_\Lambda)} \prod_{x \in \Lambda} (d\theta_x / 2\pi) \tag{3.4}$$

where $Z_\Lambda(\beta, h)$ is the partition function chosen such that the measure (3.4) is a probability measure.

The expectation in this measure is denoted $\langle \cdot \rangle_\Lambda^{XY}(\beta, h)$, and $\langle \cdot \rangle_\Lambda^{XY}(\beta) = \langle \cdot \rangle_\Lambda^{XY}(\beta, h = 0)$.

For a large class of boundary conditions (e.g., free, periodic, . . .) the thermodynamic limit

$$\left\langle \prod_{x \in A} S_x^{\alpha_x} \right\rangle^{XY}(\beta, h) = \lim_{\Lambda \to \mathbb{L}} \left\langle \prod_{x \in A} S_x^{\alpha_x} \right\rangle_\Lambda (\beta, h)$$

exists for arbitrary $A \subset \mathbb{L}$ and arbitrary $\{\alpha_x = 1, 2\}_{x \in A}$. For $h \neq 0$, $\langle \cdot \rangle^{XY}(\beta, h)$ is the unique translation-invariant equilibrium state of the rotator model[17]; moreover, for all β for which $\lim_{h \to 0} \langle S_x^1 \rangle^{XY}(\beta, h) = 0$, $\langle \cdot \rangle^{XY}(\beta)$ is the unique translation-invariant equilibrium state.[18] Thus, for $\beta = 2$, $\langle \cdot \rangle^{XY}(\beta)$ is unique for all $\beta < \infty$, by Mermin's theorem.[8] Let

$$m(\beta, h) = \lim_{\lambda \to \infty} -(1/\lambda) \log \langle \mathbf{S}_0 ; \mathbf{S}_{\lambda e} \rangle^{XY}(\beta, h) \tag{3.5}$$

where e is a unit lattice vector in $\mathbb{Z}^\nu$, and let

$$\chi(\beta, h) = \sum_{x \in \mathbb{L}} \langle \mathbf{S}_0 ; \mathbf{S}_x \rangle^{XY}(\beta, h) \tag{3.6}$$

Here

$$\langle \mathbf{S}_0 ; \mathbf{S}_x \rangle^{XY}(\beta, h) = \langle \mathbf{S}_0 \cdot \mathbf{S}_x \rangle^{XY}(\beta, h) - |\langle \mathbf{S}_0 \rangle^{XY}(\beta, h)|^2 \tag{3.7}$$

$m(\beta, h)$ is the inverse correlation length (= mass) and $\chi(\beta, h)$ the susceptibility. If $m(\beta, h) > 0$ then $\chi(\beta, h) < \infty$.

The following results are well known.

(i) For real $h \neq 0$ $m(\beta,h) > 0$, and $m(\beta,h) = O(h)$ if $m(\beta) \equiv m(\beta,0) = 0$. Moreover $m(\beta,h)$ and $m(\beta)$ are decreasing in β. Therefore, defining $\underline{\beta}_c$ by

$$\underline{\beta}_c = \inf\{\beta : m(\beta) = 0\} \tag{3.8}$$

we have that

$$m(\beta) = 0 \qquad \text{for all } \beta > \underline{\beta}_c \tag{3.9}$$

(One sees by a standard high-temperature expansion that $\underline{\beta}_c > 0$, for all dimensions ν.) For proofs see Ref. 19; see also Ref. 21, Theorem III.1.

(ii) For $\nu \geqslant 3$ there exists $\bar{\beta}_c < \infty$ such that for $\beta > \bar{\beta}_c$

$$\lim_{h \to 0} \langle S_x^1 \rangle^{XY}(\beta, h) \neq 0 \tag{3.10}$$

i.e., there exists a phase transition with order parameter at $\beta = \bar{\beta}_c$, and for $\beta > \bar{\beta}_c$ there is at least a full circle of pure phases and there exists a Goldstone excitation. If

$$\langle \mathbf{S}_0, \mathbf{S}_x \rangle^{XY}(\beta) \approx |x|^{-(\nu-2+\eta)} \qquad \text{as } |x| \to \infty$$

then

$$\eta \geqslant 0 \tag{3.11}$$

For

$$\beta > \bar{\beta}_c, \qquad m(\beta) = 0 \quad \text{and} \quad \chi(\beta) = \infty \tag{3.12}$$

Finally,

$$\bar{\beta}_c \geqslant \underline{\beta}_c \tag{3.13}$$

These results are proven in Ref. 20. (For $\nu = 2$, $\bar{\beta}_c = \infty$, by Mermin's theorem.)

(iii) As noted in Ref. 21, the Lebowitz inequalities

$$\langle S_{x_1}^\alpha ; S_{x_2}^\alpha ; S_{x_3}^\alpha ; S_{x_4}^\alpha \rangle^{XY}(\beta) \leqslant 0, \qquad \alpha = 1, 2 \tag{3.14}$$

and the inequalities

$$\left\langle S_{x_1}^1 S_{x_2}^1 ; S_{x_3}^2 S_{x_4}^2 \right\rangle^{XY}(\beta) \leqslant 0 \tag{3.15}$$

(proven in Refs. 22 and 23) together with Ginibre's inequalities[24] permit one to extend a remarkable result for the Ising model due to Glimm and Jaffe[24] to the classical XY model:

Theorem 3.1. For $\beta > \underline{\beta}_c$, $m(\beta) = 0$,

$$\lim_{\beta \uparrow \beta_c} m(\beta) = 0, \qquad \lim_{\beta \uparrow \beta_c} \chi(\beta) = \infty$$

For $\nu \geqslant 3$, the expectation

$$\langle \, \rangle^{XY}(\underline{\beta}_c) = \lim_{\beta \uparrow \beta_c} \langle \, \rangle^{XY}(\beta)$$

is clustering (i.e., extremal).

Remarks. As pointed out by Glimm and Jaffe[25] (see also Refs. 26 and 27) Theorem 3.1 proves the existence of a critical point and of a critical XY model with 0 mass, ∞ susceptibility, but *no* long-range order in $\nu \geqslant 3$ dimensions. Since $\lim_{\beta \uparrow \beta_c} \chi(\beta) = \infty$, $\eta(\underline{\beta}_c)$ [see (3.11)] satisfies[27]

$$0 \leqslant \eta(\underline{\beta}_c) \leqslant 2 \tag{3.16}$$

Simplifying somewhat one can say that, for $h \neq 0$ or $\nu \geqslant 3$, the qualitative understanding of the classical XY model is quite perfect. *New rigorous results must therefore be looked for at $h = 0$ in $\nu = 2$ dimensions* ($\nu = 1$ being trivial). Much of this paper has grown out of an attempt to prove the following conjecture.

Conjecture 3.2 (see Refs. 1, 2, and 28). For $\nu = 2$, $\underline{\beta}_c < \infty$ [so that $m(\beta) = 0$ for sufficiently large $\beta < \infty$].

This conjecture would imply that the two-dimensional XY model has a *phase transition without order parameter and an interval* $[\underline{\beta}_c, \infty)$ *of critical points*. Although a complete proof of this conjecture has so far eluded our abilities, we hope that this paper uncovers some basic mechanisms (both physical and mathematical) that should, in principle, almost suffice to prove it. Of course, Conjecture 3.2 is predicted by physical reasoning [1,2] and renormalization group calculations.[28] A complete proof might shed new light on that method. Next, we recall an important inequality proved in Ref. 29.

Theorem 3.3 (McBryan–Spencer Upper Bound). For arbitrary $\epsilon > 0$ there exists a constant $K_\epsilon < \infty$ such that the spin–spin correlation of the two-dimensional XY model satisfies

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_x \rangle^{XY}(\beta) \leqslant K_\epsilon (1 + |x|)^{-1/[(2\pi + \epsilon)\beta]}$$

Next, we state a lower bound for $\underline{\beta}_c$.

Theorem 3.4. For the two-dimensional, classical XY model

$$\underline{\beta}_c > 0.67 \tag{3.17}$$

and

$$m(\beta) > 0 \qquad \text{for all } \beta < 0.67 \tag{3.18}$$

Remarks. (1) The *proof* follows from an improved version of Refs. 30 and 31 and will be given elsewhere. We also remark that a straightforward combination of Theorems 3.1 and 3.3 yields $\underline{\beta}_c \geqslant 1/4\pi$, i.e., $\sim 1/8$ of the lower bound (3.17). The mean field bound is $\overline{\beta}_c = 1/2$.

(2) The exact value of $\underline{\beta}_c$ is conjectured to be ≈ 1, so that (3.17) is as accurate as can be expected from an expansion method (our proof of Theorem 3.4 involves an expansion in the spirit of Ref. 30; see also Ref. 31).

(3) Consider the N-vector (N-component, classical spin) models, $N = 1, 2, 3, \ldots,$ with Hamiltonian given by (3.3), and $\mathbf{S} \in S^{N-1}$ ($N = 1$ is the Ising and $N = 2$ the classical XY model). In Ref. 16 we have proven that for $\nu = 2$

$$m_N(\beta) \leqslant c_1 \exp\left[-c_2(\beta/N)\right] \tag{3.19}$$

for some positive constants c_1, c_2. Using methods of Ref. 30 we have been able to show that

$$\underline{\beta}_c(N) > N/2\nu \tag{3.20}$$

Based on approximate calculations it has been conjectured that $\underline{\beta}_c(N) = \infty$, for $\nu = 2$ and $N \geqslant 3$; see Ref. 32. There is *no* proof of this!

This concludes our list of rigorous results for the classical XY model.

As noted in Ref. 3, it is useful to compare the two-dimensional XY model with the *two-dimensional Villain model*, for which a proof of Conjecture 3.2 might be a little more accessible and which is isomorphic to a two-dimensional lattice Coulomb gas. Recall that for the XY model with $h = 0$

$$e^{-\beta H(\theta_\Lambda)} = \prod_{xy \subset \Lambda} e^{\beta \cos(\theta_x - \theta_y)}; \qquad \text{see (3.3)} \tag{3.21}$$

The Villain model is obtained by replacing $r_\beta(\theta) = \exp[\beta \cos\theta]$ in (3.21) and (3.4) by

$$v_\beta(\theta) = \sum_{n \in \mathbb{Z}} \exp\left[-(\beta/2)(\theta + 2\pi n)^2\right] \tag{3.22}$$

When necessary we distinguish the XY and the Villain model by adding a superscript v when considering the Villain model.

Note that $v_\beta(\theta - \theta')$ is (up to a constant factor) the integral kernel of the operator $\exp[(1/2\beta)\Delta]$, where Δ is the Laplacean on S^1 (= Laplacean on $[0, 2\pi]$ with periodic boundary conditions at 0 and 2π). Thus, using the

Trotter product formula,

$$v_\beta(\theta-\theta') = \lim_{N\to\infty} C(N,\beta)\int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_N \prod_{j=0}^{N} r_{\beta N}(\theta_j-\theta_{j+1})$$

with $\theta_0 = \theta$ and $\theta_{N+1} = \theta'$; see, e.g., Ref. 33.

One may therefore view the Villain model as a limit of rotator models in which each link $xy \subset \mathbb{Z}^\nu$ is occupied by N classical, two-component spins interacting with their nearest neighbors. This implies that the Ginibre inequalities[24] hold for the Villain model, as noted by Bellissard.[34] (There is an independent proof involving duality transformation; see Section 3.2.) Moreover, the Lee–Yang theorem,[35] the Lebowitz inequalities (3.14), and inequality (3.15)[22,23] clearly remain true, as well. Finally, the method of proof of Theorem 3.3 (McBryan–Spencer upper bound) can also be applied to the Villain model. (In fact, the proof[29] of Theorem 3.3 for the Villain model is simpler, and one can set $\epsilon = 0$, $K_{\epsilon=0} = K$ in Theorem 3.3, as the reader easily checks. See also Section 5.) In conclusion, *all results summarized for the two-dimensional, classical XY model, in particular* (*i*) *and* (*iii*), *extend to the two-dimensional Villain model.* Among these we have the following theorem.

Theorem 3.5. In the two-dimensional Villain model

$$\langle e^{i(\theta_0-\theta_x)}\rangle^{v}(\beta) \leqslant K(1+|x|)^{-(1/2\pi\beta)}$$

$$m^v(\beta) > 0 \qquad \text{for all } \beta < \underline{\beta}_c^v \qquad \text{with } \underline{\beta}_c^v \geqslant \frac{1}{2\pi}$$

(see Ref. 27). As in the XY model we make the following conjecture.

Conjecture 3.2^v. In the two-dimensional Villain model, $\underline{\beta}_c^v < \infty$.

Remark. Heuristic arguments based on comparing the dual (Fourier transformed) Villain model with the dual XY model (see Section 3.2) suggest that Conjecture 3.2^v implies Conjecture 3.2; see also Ref. 28. We do not elaborate on this point, but emphasize that the machinery developed in this paper for approximate Villain models can also be applied to approximate XY models, so that the two conjectures ought to have closely related proofs.

3.2. The Dual XY and Villain Models

In this section we use Fourier transformation in the angles $\{\theta_x\}$ in order to replace the XY and Villain models in two dimensions by models of classical, one-component (Ising-type) spins with values in the integers. This is the well-known Kramers–Wannier duality transformation. We only present results. For proofs see Refs. 3 and 28.

Let $\hat{r}_\beta(n)$, $\hat{v}_\beta(n) = c_\beta \exp[-(\frac{1}{2\beta})n^2]$ denote the Fourier coefficients of the functions $r_\beta(\theta)$, $v_\beta(\theta)$, respectively.

In two dimensions, define equilibrium expectations of the dual XY and the dual Villain model in a finite volume Λ (with 0 boundary conditions) by the following measures:

$$d\mu^{\hat{r}}(\phi_\Lambda) = \left[Z_\Lambda^{\hat{r}}(\beta)\right]^{-1} \prod_{xy\subset\Lambda} \hat{r}_\beta(\phi(x)-\phi(y)) \prod_{x\in\Lambda} d\rho[\phi(x)] \tag{3.23}$$

$$d\mu^{\hat{v}}(\phi_\Lambda) = \left[Z_\Lambda^{\hat{v}}(\beta)\right]^{-1} \prod_{xy\subset\Lambda} \exp\left\{-(\tfrac{1}{2\beta})\left[\phi(x)-\phi(y)\right]^2\right\} \prod_{x\in\Lambda} d\rho\left[\phi(x)\right] \tag{3.24}$$

where $d\rho(\phi) = [\sum_{m\in\mathbb{Z}}\delta(\phi - m)]\,d\phi$ and $Z_\Lambda^{\hat{r}}(\beta)$, $Z_\Lambda^{\hat{v}}(\beta)$ are the obvious normalization factors. Let $\langle\cdot\rangle_\Lambda^{\hat{r}}(\beta)$, $\langle\cdot\rangle_\Lambda^{\hat{v}}(\beta)$ denote the expectations determined by $d\mu^{\hat{r}}$, $d\mu^{\hat{v}}$, respectively. Let x be the site $(n,0) = ne_1 \in \mathbb{Z}^2$. We define

$$A^f(q;0,x) = \prod_{m=0}^{n-1} \frac{f_\beta(\theta_{(m,1)} - \theta_{(m,0)} + 2\pi q)}{f_\beta(\theta_{(m,1)} - \theta_{(m,0)})} \tag{3.25}$$

where $f_\beta = r_\beta$ or v_β and $q \in (0,1)$, and

$$A^{\hat{f}}(k;0,x) = \prod_{m=0}^{n-1} \frac{\hat{f}_\beta(\phi(m,1) - \phi(m,0) + k)}{\hat{f}_\beta(\phi(m,1) - \phi(m,0))} \tag{3.26}$$

where $\hat{f}_\beta = \hat{r}_\beta$ or $\hat{v}_\beta$ and $k \in \mathbb{Z}$.

Note that

$$A^{\hat{v}}(k;0,x) = \left(\prod_{m=0}^{n-1} \exp\left\{-(1/\beta)\left[\phi(m,1) - \phi(m,0)\right]k\right\}\right) \times \exp\left[-(\tfrac{1}{2\beta})k^2|x|\right] \tag{3.27}$$

Let $\Lambda' = \Lambda'(\Lambda)$ the region corresponding to Λ in the dual lattice. From Refs. 28 and 21 we have the following theorem.

Theorem 3.6

(1) $\langle e^{ik(\theta_0-\theta_x)}\rangle_\Lambda(\beta) = \langle A^{\hat{r}}(k;0,x)\rangle_{\Lambda'}(\beta)$

(2) $\langle e^{ik(\theta_0-\theta_x)}\rangle_\Lambda^{v}(\beta)$

$$= \left\langle \prod_{m=0}^{n-1} \exp\left\{-(1/\beta)\left[\phi(m,1) - \phi(m,0)\right]k\right\}\right\rangle_{\Lambda'}^{\hat{v}}(\beta)\, e^{[-(1/2\beta)k^2|x|]}$$

(3) $\langle A^r(q;0,x)\rangle_\Lambda(\beta) = \left\langle \exp\left\{i2\pi q\left[\phi(0) - \phi(x)\right]\right\}\right\rangle_{\Lambda'}^{\hat{r}}(\beta)$

(4) $\langle A^v(q;0,x)\rangle_\Lambda^{\hat{v}}(\beta) = \left\langle \exp\left\{i2\pi q\left[\phi(0) - \phi(x)\right]\right\}\right\rangle_{\Lambda'}^{\hat{v}}(\beta)$

Remarks. For $k = 1$ one has of course

$$\langle e^{i(\theta_0-\theta_x)}\rangle_\Lambda(\beta) = \langle \mathbf{S}_0 \cdot \mathbf{S}_x\rangle_\Lambda(\beta)$$

For $q \in \mathbb{Z}$, $\langle e^{i2\pi q[\phi(0)-\phi(x)]}\rangle_{\Lambda}(\beta) = 1$.

In principle, Theorem 3.6 can be extended to arbitrary correlation functions. This establishes *an isomorphism between the XY model and the* $\hat{r}$ *model defined in* (3.23) *and between the Villain model and the* $\hat{v}$ *model defined in* (3.24), *in two dimensions.*

Proof. The proof of Theorem 3.6 follows by Fourier transformation in the variables $\{\theta_x - \theta_y\colon xy$ nearest neighbors in $\Lambda\}$ and application of the lattice version of Poincaré's lemma,

$$*\partial * k = 0 \Rightarrow k = *\partial\phi \tag{3.28}$$

where k is a lattice 1-form (lattice vector field), and ϕ is a lattice $(\nu-2)$-form. Thus, in $\nu = 2$ dimensions, ϕ is a scalar, i.e., a function on the lattice. For $\nu = 3$, an analog of Theorem 3.6 holds: consider, e.g., the three-dimensional Villain model. In this case, ϕ is a lattice vector field, the components, ϕ_{xy} (xy nearest neighbors), take values in $\mathbb{Z}$. Thus the dual of the three-dimensional Villain model is an *Abelian lattice gauge theory.*

Next, we discuss another version of Conjectures 3.2 and 3.2^v which involves the dual two-point correlations, i.e., the two-point functions of the $\hat{r}$ and $\hat{v}$ model introduced in Theorem 3.6(3) and (4). Suppose β is very small. Then one deduces from (3.23), (3.24), and the small-β behavior of $\hat{r}_\beta$ and $\hat{v}_\beta$ that, in the presence of 0 ($\equiv$ free) boundary conditions at $\partial\Lambda$,

$$0 \leqslant \langle\phi(0)\phi(x)\rangle_\Lambda^{\hat{r}(\hat{v})}(\beta) \leqslant O\big(\exp[-m(\beta)|x|]\big) \tag{3.29}$$

and

$$\left|\langle e^{iq\phi(0)}; e^{-iq\phi(x)}\rangle_\Lambda^{\hat{r}(\hat{v})}(\beta)\right| \leqslant O\big(\exp[-m(\beta)|x|]\big) \tag{3.30}$$

uniformly in Λ, for all $q \in (0, 1)$.

The proof is based on a straightforward Peierls contour expansion—in the style of Refs. 36 and 4, but much simpler. Incidentally, (3.29) implies (3.30) if the model satisfies FKG inequalities,[37] which is obvious for the $\hat{v}$ model.

The Peierls contour expansion also shows that

$$\left|\langle e^{iq\phi(0)}\rangle^{\hat{r}(\hat{v})}(\beta)\right| \geqslant M_q(\beta) > 0 \tag{3.31}$$

uniformly in Λ, for sufficiently small β. (Related results and proofs may be found, e.g., in Ref. 4.) In the case of the $\hat{v}$ model, (3.31) also follows directly from (3.30) and the inequality

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle_\Lambda^{\hat{v}}(\beta) \gtrsim |x|^{-q^2\beta/2\pi} \tag{3.32}$$

for Λ and $|x|$ suitably large, which we prove in Section 3.3.

Thus, in the thermodynamic limit, (3.30) and (3.31) yield

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle^{\hat{r}(\hat{v})}(\beta)\to\left|\langle e^{iq\phi(0)}\rangle^{\hat{r}(\hat{v})}(\beta)\right|^2\geqslant M_q(\beta)^2>0 \tag{3.33}$$

as $|x|\to\infty$, for all β with $0<\beta<\beta_c^{\hat{r}(\hat{v})}$.

In Section 3.3 we shall see that this temperature range corresponds to one in the two-dimensional Coulomb gas in the (Mnhc) grand canonical ensemble (see Section 2.2) with exponential Debye screening.

Now we consider the $\beta\gg 1$ regime: for $\beta\gg 1$ the expectation $\langle\cdot\rangle^{\hat{v}}(\beta)$ is, heuristically speaking, very close to the Gaussian expectation with mean 0 and covariance $(-\Delta)^{-1}$. To see this, rescale $\phi\to\phi'=\beta^{-1/2}\phi\in\beta^{-1/2}\mathbb{Z}$ and observe that $d\rho(\beta^{1/2}\phi')\to d\phi$ (the Lebesgue measure), as $\beta\to\infty$ (on C_0^0). For the Gaussian, $\langle\cdot\rangle_C$,

$$\langle e^{iq\beta^{1/2}[\phi'(0)-\phi'(x)]}\rangle_C\approx|x|^{-q^2\beta/2\pi} \tag{3.34}$$

as $|x|\to\infty$, in contrast to (3.33).

Thus we propose the following

Conjecture 3.7. For the $\hat{r}$ and $\hat{v}$ models there exist critical temperatures $\beta_c^{\hat{r}}<\infty$, $\beta_c^{\hat{v}}<\infty$ such that, for $\beta>\beta_c^{\hat{r}}(\beta_c^{\hat{v}})$,

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle^{\hat{r}(\hat{v})}(\beta)\to 0 \qquad \text{as } |x|\to\infty \tag{3.35}$$

Remarks. (1) Clearly Conjecture 3.7 is related to Conjectures 3.2 and 3.2^v. We believe that a constructive proof of Conjecture $3.\hat{7}$ will also yield a proof of Conjectures 3.2 and 3.2^v and that $\beta_c^{\hat{r}}=\underline{\beta}_c$, $\beta_c^{\hat{v}}=\underline{\beta}_c^v$, but there is no rigorous proof of these equations. See also Section 5.

Conjecture 3.7^v appears to be somewhat easier to analyze than the "dual" Conjecture 3.2^v. Sections 4, 5, and 6 are devoted to working up some ideas and methods that should enable one to prove Conjecture $3.7^{v(r)}$.

(2) We emphasize that Conjecture $3.7^{r(v)}$ really says that the $\hat{r}(\hat{v})$ model has a *phase transition with order parameter*: for $\beta<\beta_c^{\hat{r}(\hat{v})}$,

$$\left|\langle e^{iq\phi(0)}\rangle^{\hat{r}(\hat{v})}(\beta)\right|\geqslant M_q(\beta)>0$$

whereas for $\beta>\beta_c^{\hat{r}(\hat{v})}$

$$\langle e^{iq\phi(0)}\rangle^{\hat{r}(\hat{v})}(\beta)=0$$

by (3.35), i.e., $\langle e^{iq\phi(0)}\rangle^{\hat{r}(\hat{v})}(\beta)$ is the *order parameter*. Phase transitions with order parameter often tend to be easier to handle than ones without.

(3) The reader familiar with a recent paper of Mack and Petkova[38] should note that their modification of the $SU(2)$ lattice gauge theory has an analog in the two-dimensional XY model. An adaptation of their estimates shows that, in the modified two-dimensional XY model, $\langle e^{iq[\phi(0)-\phi(x)]}\rangle(\beta)\to 0$, as $|x|\to\infty$, exponentially fast, for large enough $\beta<\infty$. This is clearly

incompatible with (3.31) and suggests that the Mack–Petkova modification has a serious effect on the large-β behavior of those lattice models.

Next, we recall an *isomorphism* between the $\hat{v}$ model and the two-dimensional *lattice Coulomb gas* [in an (Mg) ensemble; see Section 2.2].

Note that

$$d\rho(\phi) = \left[\sum_{m \in \mathbb{Z}} \delta(\phi - m)\right] d\phi$$
$$= \left[\sum_{q \in \mathbb{Z}} e^{i2\pi\phi q}\right] d\phi,$$

(in distribution sense) by the Poisson summation formula. Clearly

$$\sum_{q \in \mathbb{Z}} e^{i2\pi q\phi} = \int e^{iq\phi}\, d\lambda(q)$$

with

$$d\lambda(q) = \left[\sum_{m \in \mathbb{Z}} \delta(q - 2\pi m)\right] dq \tag{3.36}$$

We now recall that $\hat{v}_\beta(\phi) = \exp[-(1/2\beta)\phi^2]$, so that when $d\rho$ is replaced by $d\phi$, $\phi(\cdot)$ is a Gaussian process with mean 0 and covariance $(-\Delta)^{-1}$. Comparison with the definition of the (Mg) ensemble of Section 2.2, see (2.21)–(2.23), now exhibits the isomorphism. In the charge variables, $q_\Lambda = \{q_x\}_{x \in \Lambda}$, the equilibrium measure of the two-dimensional Villain model is given by

$$Z_\Lambda(\beta)^{-1} \exp\left[-(\beta/2)4\pi^2 \sum_{x,y \text{ in } \Lambda} q_x C_\Lambda(x, y) q_y\right] \tag{3.37}$$

where $C_\Lambda(x, y)$ is the kernel of $(-\Delta_\Lambda)^{-1}$, Δ_Λ the finite difference Laplacean with 0-Dirichlet data at $\partial\Lambda$, and $Z_\Lambda(\beta)$ is the partition function. The proof of (3.37) follows directly from (3.36) and (2.23).

It is now clear that $\langle e^{iq[\phi(0)-\phi(x)]}\rangle^{\hat{v}}(\beta)$ is a fractional charge two-point correlation, with q the fraction of the charge of the test particles and the charge of the background particles.

3.3. Comparison of the $\hat{v}$ Model with the Two-Dimensional Coulomb Gas in the (Mnhc) Grand Canonical Ensemble

We show here that the $\hat{v}$ model is the $z = \infty$ limit of the two-dimensional Coulomb gas in the (Mnhc) grand canonical ensemble with equilibrium expectation $\langle \cdot \rangle'_\Lambda(\beta, z)$ given by the measure

$$\Xi_\Lambda(\beta, z)^{-1} \prod_{x \in \Lambda} \exp\{z \cos[2\pi\phi(x)]\}\, d\mu_{\beta C_\Lambda}(\phi) \tag{3.38}$$

which is obtained from the expectation $\langle\cdot\rangle_\Lambda(\beta',z)$ with 0 ($\equiv$ free) boundary conditions at $\partial\Lambda$, defined in (2.16) (I) by rescaling: $\phi(x)\to\phi'(x) = 2\pi\phi(x)$, and setting

$$\beta' = 4\pi^2\beta \tag{3.39}$$

It is shown in Ref. 11 [see also Theorem 2.4(2), Section 2.1] that

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle_\Lambda(\beta,z) \text{ is monotone increasing in } z \text{ and } \Lambda^{-1}. \tag{3.40}$$

As remarked in Section 2.1, monotonicity permits one to pass to the thermodynamic limit, $\Lambda = \mathbb{Z}^2$, and conclude that $\langle e^{iq[\phi(0)-\phi(x)]}\rangle'(\beta,z)$ is monotone increasing in z.

When Λ is bounded

$$\lim_{z\to\infty}\langle e^{iq[\phi(0)-\phi(x)]}\rangle'_\Lambda(\beta,z) = \langle e^{iq[\phi(0)-\phi(x)]}\rangle_\Lambda^{\hat{v}}(\beta) \tag{3.41}$$

since

$$(2\pi z)^{1/2}e^{z[\cos(2\pi\phi)-1]}\to\sum_{m\in\mathbb{Z}}\delta(\phi-m).$$

By (3.40) and (3.41),

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle^{\hat{v}}(\beta) \geqslant \lim_{z\to\infty}\langle e^{iq[\phi(0)-\phi(x)]}\rangle'(\beta,z) \tag{3.42}$$

$$\geqslant \langle e^{iq[\phi(0)-\phi(x)]}\rangle'(\beta,z) \tag{3.43}$$

$$\geqslant \langle e^{iq[\phi(0)-\phi(x)]}\rangle'(\beta,z=0)$$

$$= \langle e^{iq[\phi(0)-\phi(x)]}\rangle_{\beta C}$$

$$= O\big(|x|^{-q^2\beta/2\pi}\big) \qquad \text{as } |x|\to\infty \tag{3.44}$$

This proves inequality (3.32) of Section 3.2. Suppose now that Conjecture 3.7^{v} holds, i.e., for $\beta > \beta_c^{\hat{v}}$ with $\beta_c^{\hat{v}}$ finite,

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle^{\hat{v}}(\beta)\to 0 \qquad \text{as } |x|\to\infty$$

By (3.43), this implies that, for $\beta > \beta_c(z)$ with $\beta_c(z) \leqslant \beta_c^{\hat{v}}$, for all z,

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle'(\beta,z)\to 0 \qquad \text{as } |x|\to\infty$$

and

$$\langle e^{iq\phi(0)}\rangle'(\beta,z) = 0 \tag{3.45}$$

For $\beta \ll 1$ and $z \leqslant \infty$ suitably large, Brydges has shown[4] that $\langle e^{iq\phi(0)}; e^{-iq\phi(x)}\rangle'(\beta,z)$ decays exponentially when $|x|\to\infty$, provided $\langle\cdot\rangle'(\beta,z)$ is a 0-boundary condition thermodynamic limit. (In Section 4 we show that this is *false* for the "Gaussian" boundary conditions considered in Sections 2.1 and 2.2.)

Brydges' result and (3.44) prove that, for $\beta \ll 1$ and $z \leqslant \infty$ suitably large,

$$\left|\langle e^{iq\phi(0)}\rangle'(\beta, z)\right| > 0 \tag{3.46}$$

Thus we have the following theorem.

Theorem 3.8. Suppose that Conjecture 3.7^v holds. Then, for all sufficiently large activities z, the two-dimensional Coulomb gas in the (Mnhc) grand canonical ensemble (with 0 boundary condition) has a phase transition with order parameter, $\langle e^{iq\phi(0)}\rangle'(\beta, z)$, from a high temperature (small β) phase with Debye screening characterized by exponential clustering and (3.46) to a low-temperature (dipolar) phase characterized by slow decay of the fractional charge two-point correlation and (3.45). Moreover

$$\beta_c(z) \leqslant \hat{\beta}_c^v \qquad \text{for all } z \leqslant \infty \tag{3.47}$$

Thus, a problem somewhat easier than a proof of Conjecture 3.7^v is to prove that for $z > 0$ small enough there exists $\beta_c(z) < \infty$ such that, for all $\beta > \beta_c(z)$, (3.45) holds. We present arguments in the direction of a proof of this, based on relating, the two-dimensional Coulomb gas at low temperatures to a two-dimensional *Coulomb-dipole gas* for which (3.45) is relatively easy to prove. See Sections 4 and 5. Finally, we remark that the $\hat{r}$ model can also be related to a Coulomb type gas which can be studied by the same methods as the (Mnhc) gas.

4. THE STATISTICAL MECHANICS OF LATTICE COULOMB GASES

In this section we review some rigorous results and prove new ones, all concerning the two- (and higher-) dimensional lattice Coulomb gas in the grand canonical ensembles [the (Mnhc) or (Mhc) ensemble introduced in Section 2.2].

4.1. On Screening Properties and the Phase Diagram of Coulomb Monopole Gases

In the ϕ representation the equilibrium expectations, $\langle\,\cdot\,\rangle_\Lambda(\beta, z)$ and $\langle\,\cdot\,\rangle_\Lambda^{\mathrm{hc}}(\beta, z)$, of the (Mnhc) or (Mhc) ensemble are given by the measures

$$\text{(Mnhc)} \qquad \Xi_\Lambda(\beta, z)^{-1} \prod_{x\in\Lambda} e^{z\cos\phi(x)}\, d\mu_{\beta C_\Lambda}(\phi)$$

$$\text{(Mhc)} \qquad \Xi_\Lambda^{\mathrm{hc}}(\beta, z)^{-1} \prod_{x\in\Lambda} \left[1 + z\cos\phi(x)\right] d\mu_{\beta C_\Lambda}(\phi) \tag{4.1}$$

with $d\mu_{\beta C_\Lambda}$ the Gaussian measure with mean 0 and covariance βC_Λ, where $C_\Lambda(x, y)$ is the Coulomb potential with *either* free boundary conditions (b.c.) at $\partial\Lambda$, i.e., $C_\Lambda(x, y) = \chi_\Lambda(x)C(x - y)\chi_\Lambda(y)$, $C(x)$ the lattice Coulomb potential, *or* 0-Dirichlet b.c. at $\partial\Lambda$.

In the first case we say $\langle \cdot \rangle_\Lambda^{(\mathrm{hc})}(\beta, z)$ has free, in the second case that it has 0 b.c. Physically, free b.c. correspond to confining the gas in the interior of *perfectly insulating walls*, whereas 0 b.c. correspond to *perfectly conducting walls*. (Clearly there are intermediate b.c.)

We now show that in two dimensions the difference between free and 0 b.c. is reflected in very different screening properties of the Coulomb gas, *even in the thermodynamic limit*. To see this we consider fractional charge one- and two-point correlations (see Sections 3.2, 3.3).

Theorem 4.1. Let $z > 0$, and in the case of the (Mhc) ensemble $z < 1$ [so that $1 + z\cos\phi(x) > 0$].

(1) In arbitrary dimension $\nu \geqslant 2$ and for 0 b.c., there is exponential Debye screening if β is small enough. (For $\nu \geqslant 3$, this is true for arbitrary β and small z.) The fractional charge one-point correlation is nonzero, the connected (truncated) fractional charge two-point correlation decays exponentially.

(2) In two dimensions and for free b.c. and all $q \in (0, 1)$, all $z \geqslant 0$,

$$\langle e^{iq\phi(0)} \rangle_\Lambda^{(\mathrm{hc})}(\beta, z) = 0 \tag{4.2}$$

$$\langle e^{iq[\phi(0) - \phi(x)]} \rangle_\Lambda^{(\mathrm{hc})}(\beta, z) \geqslant O(|x|^{-q^2\beta/2\pi}) \tag{4.3}$$

as $|x| \to \infty$, for arbitrary $\Lambda \subseteq \mathbb{Z}^2$ and all β.

Remarks. There are heuristic reasons to expect that for $\nu \geqslant 3$ Theorem 4.1(1) is true for all β and that in the thermodynamic limit 0 and free b.c. coincide. As remarked in Section 3.3, the proof of (1) for the (Mnhc) ensemble is due to Brydges.[(4)] His proof extends to the (Mhc) ensemble, for $\beta \ll 1$, $0 < z < 1$. [We thank D. Brydges for checking some details in his proof for the (Mhc) ensemble.]

Theorem 4.1(2) shows that in the *two-dimensional, free b.c.* equilibrium states *fractional charges are not screened*, even in the thermodynamic limit and for arbitrary β. The classical Goldstone picture based on the behavior of the functions $z\cos\phi(x)$ or $\ln[1 + z\cos\phi(x)]$ $(z < 1)$ and the Peierls argument suggest that for $\beta \ll 1$

$$\langle e^{iq[\phi(0) - \phi(x)]} \rangle^{(\mathrm{hc})}(\beta, z) \geqslant M_q(\beta, z) > 0 \tag{4.4}$$

for all x (i.e., there is "long-range order") even for free b.c. Thus, for $\beta \ll 1$, the free b.c. expectation in the thermodynamic limit is presumably not clustering. A proof of this is expected to follow from the methods of Refs. 39 and 4, but we have not checked the details.

Goldstone and Peierls suggest that the free b.c. state in the thermodynamic limit is of the form

$$\langle\,\cdot\,\rangle^{(\mathrm{hc})}(\beta,z)=\lim_{N\to\infty}\sum_{m=-N}^{N}c_m^{(N)}\langle\,\cdot\,\rangle_m^{(\mathrm{hc})}(\beta,z) \tag{4.5}$$

with $\langle\phi(0)\rangle_m^{(\mathrm{hc})}(\beta,z)=2\pi m$, and

$$\begin{aligned}&\langle F(\phi-m)\rangle_m^{(\mathrm{hc})}(\beta,z)=\langle F(\phi)\rangle_0^{(\mathrm{hc})}(\beta,z)\\&\langle\,\cdot\,\rangle_0^{(\mathrm{hc})}(\beta,z)\qquad\text{identical to the 0 b.c. state}\end{aligned} \tag{4.6}$$

From (4.6) it follows that

$$\left\langle e^{iq\phi(0)}\right\rangle_m^{(\mathrm{hc})}(\beta,z)=\mathrm{const}\,e^{i2\pi qm} \tag{4.7}$$

Moreover, $\langle e^{iq\phi(0)}\rangle^{(\mathrm{hc})}(\beta,z)=0$. Hence

$$c_m^{(N)}=1/2N+1\qquad\text{in (4.5)} \tag{4.8}$$

Since the right-hand side of (4.7) is not real, for $m\neq 0$ and because of (4.8), the decomposition (4.5) does not represent a decomposition of the equilibrium state of the Coulomb gas with free b.c. into physical, extremal equilibrium states. The states $\langle\,\cdot\,\rangle_m^{(\mathrm{hc})}(\beta,z)$ are unphysical states of the Coulomb gas (in the q representation) characterized by complex boundary conditions. Thus the free b.c. state is physically different from the 0 b.c. state!

Proof of Theorem 4.1. We have already commented on the proof of (1). Moreover, inequality (4.3) is (3.46) of Section 3.3. We are left with proving (4.2). This is based on the following lemma.

Lemma 4.2. Let $\phi(\rho)=\sum\phi(x)\rho(x)$, $\rho\in l_1(\mathbb{Z}^2)$. Then

$$\left\langle e^{i\phi(\rho)}\right\rangle_{\beta C}=\begin{cases}\exp\left[(-\beta/2)\sum_{x,y}\rho(x)C(x-y)\rho(y)\right] & \text{if }\sum\rho(x)=0\\ 0 & \text{if }\sum\rho(x)\neq 0\end{cases}$$

[Here $C(x-y)\approx(1/2\pi)\ln|x-y|$, as $|x-y|\to\infty$, is the two-dimensional lattice Coulomb potential.]

Proof. The proof of Lemma 4.2 follows by noting that $C(x)=\lim_{\epsilon\downarrow 0}[C_\epsilon(x)-(1/2\pi)\ln\epsilon]$, where $C_\epsilon(x-y)$ is the integral kernel of $(-\Delta+\epsilon^2)^{-1}$, via Fourier transformation. See Ref. 10 for an exact statement and proof.

We now consider the fractional charge one-point correlation. Clearly

$$\left\langle e^{iq\phi(0)}\right\rangle_\Lambda^{(\mathrm{hc})}(\beta,z)=\Xi_\Lambda^{(\mathrm{hc})}(\beta,z)^{-1}\sum_\rho c_\rho z^{\sum_x|\rho(x)|}\left\langle e^{i[q\phi(0)+\phi(\rho)]}\right\rangle_{\beta C} \tag{4.9}$$

where $\rho(x) \in \mathbb{Z}$, for all $x \in \Lambda$, and c_ρ are combinatorial coefficients with the property that

$$\sum_\rho c_\rho z^{\sum_x |\rho(x)|} < \infty \qquad \text{for all } z > 0$$

Since $q \in (0,1)$, $q + \sum_{x \in \Lambda} \rho(x) = q + n \neq 0$, for some $n = n(\rho) \in \mathbb{Z}$. By Lemma 4.2,

$$\langle e^{i[q\phi(0) + \phi(\rho)]} \rangle_{\beta C} = \langle e^{i\phi(q \cdot \delta_0 + \rho)} \rangle_{\beta C} = 0$$

for all $\mathbb{Z}$-valued ρ on Λ. This completes the proof of (4.2); hence Theorem 4.1 is proven. ■

Next, we derive an upper bound on the inverse correlation length (mass), $m(\beta, z)$, of the ($\nu \geqslant 2$)-dimensional Coulomb gas in any translation-invariant infinite volume state, $\langle \cdot \rangle(\beta, z)$, which has screening [and with $z < 1$ in the case of the (Mhc) ensemble].

In Section 2.3, (2.29), (2.32) we have shown that under the above hypotheses

$$0 \leqslant \langle |\hat{\phi}(k)|^2 \rangle(\beta, z) = \beta \Delta(k)^{-1} - \beta^2 \Delta(k)^{-2} \langle |\hat{q}(k)|^2 \rangle(\beta, z) \tag{4.10}$$

where

$$\Delta(k)^{-1} = \left(2\nu - 2\sum_{i=1}^{\nu} \cos k^i\right)^{-1}, \qquad k \neq 0$$

is the Fourier transform of the ν-dimensional Coulomb potential. If there is screening then $\langle |\phi(k)|^2 \rangle(\beta, z)$ is analytic in k, in particular uniformly bounded. Since $\Delta(k)^{-1} = O(k^{-2})$ near $k = 0$,

$$\langle |\hat{q}(k)|^2 \rangle(\beta, z) = k^2 G(k) \tag{4.11}$$

where $G(k)$ is analytic in k and $G(0) = \beta^{-1}$. This is a sum rule which implies the well-known fact that $\langle q_\Lambda^2 \rangle(\beta, z)$ is $O(\partial\Lambda)$ ("abnormal fluctuations"). By Fourier transformation,

$$\sum_{x \in \mathbb{Z}^\nu} |x|^2 \langle q_0 q_x \rangle(\beta, z) = 2\nu\beta^{-1}$$

so that

$$\sum_{x \in \mathbb{Z}^\nu} |x|^2 |\langle q_0 q_x \rangle(\beta, z)| \geqslant 2\nu\beta^{-1} \tag{4.12}$$

Let $m \equiv m(\beta, z)$ denote the inverse correlation length. By reflection positivity (more precisely, the existence of a self-adjoint transfer matrix, see Appendix A, and the spectral theorem),

$$\begin{aligned} |\langle q_0 q_x \rangle(\beta, z)| &\leqslant -\langle q_0 q_e \rangle(\beta, z) e^{-(m/\sqrt{\nu})|x|} \\ &\leqslant |\langle q_0^2 \rangle(\beta, z)| e^{-(m/\sqrt{\nu})|x|} \end{aligned} \tag{4.13}$$

(where e is some lattice unit vector).

The proof of (4.13), given a self-adjoint transfer matrix, is standard. The chessboard estimate[14] then yields

$$\langle q_0^2\rangle(\beta,z) \leqslant \lim_{\Lambda\uparrow\mathbb{Z}^\nu}\left[\left\langle \prod_{x\in\Lambda} q_x^2\right\rangle_\Lambda(\beta,z)\right]^{1/|\Lambda|} \tag{4.14}$$

The right-hand side of (4.14) is a thermodynamic quantity which is easy to estimate:

$$\left\langle \prod_{x\in\Lambda} q_x^2\right\rangle_\Lambda(\beta,z) \leqslant \left(c\sum_{q\in\mathbb{Z}} q^2 e^{-(\beta/4\nu)q^2}\right)^{|\Lambda|} \tag{4.15}$$

where c is some constant bounded uniformly in z. Inequality (4.15) follows by writing $\langle\prod_{x\in\Lambda}q_x^2\rangle_\Lambda(\beta,z)$ as the product of $\Xi_\Lambda(\beta,z)^{-1}$ and an unnormalized expectation. Clearly, $\Xi_\Lambda(\beta,z)>1$. The unnormalized expectation is bounded above by one where all couplings between different sites have been eliminated by means of replacing the Coulomb potential, C, by $(1/2\nu)\delta_{xy}$. Here we use the inequality $(q,Cq)\geqslant(1/2\nu)(q,q)$ so that

$$\begin{aligned}\exp\left[-\frac{\beta}{2}(q,Cq)\right] &\leqslant \exp\left[-\frac{\beta}{4\nu}(q,q)\right]\\ &= \prod_{x\in\Lambda}\exp\left[-\frac{\beta}{4\nu}q_x^2\right]\end{aligned} \tag{4.16}$$

Thus

$$\left\langle \prod_{x\in\Lambda} q_x^2\right\rangle_\Lambda(\beta,z)^{1/|\Lambda|} \leqslant ae^{-\beta/4\nu} \tag{4.17}$$

for some finite constant, a, bounded uniformly in z, and all Λ. Combining (4.12)–(4.17) we find

$$2\nu\beta^{-1} \leqslant ae^{-\beta/4\nu}\sum_{x\in\mathbb{Z}^\nu}|x|^2 e^{-(m/\sqrt{\nu})|x|}$$

whence

$$\beta^{-1} \leqslant C_\nu e^{-\beta/4\nu_m-(\nu+2)}$$

or

$$m \leqslant (C_\nu\beta)^{1/\nu+2}e^{-\beta/4\nu(\nu+2)} \tag{4.18}$$

for some computable constant C_ν independent of z. Thus we have proven the following theorem.

Theorem 4.3. In the ν-dimensional Coulomb gas, the inverse correlation length $m(\beta,z)$ is bounded by

$$m(\beta,z) \leqslant O(e^{-\delta_\nu\beta}), \qquad \delta_\nu \geqslant [4\nu(\nu+2)]^{-1}$$

uniformly in z.

Remarks. Since the dual of the two-dimensional Villain model is the $z = \infty$ limit of the two-dimensional Coulomb gas in the (Mnhc) ensemble, and (4.18) is uniform in z, we obtain that the mass of the dual Villain model is bounded by $m^{\hat{v}}(\beta) \leqslant O(e^{-\beta/32})$. This inequality is comparable with the one for the two-dimensional XY model; see Section 3.1, (3.19).

[For the two-dimensional Coulomb gas we expect that

$$|\langle q_0 q_x \rangle(\beta, z)| \leqslant \min\left(e^{-c\beta} e^{-(m/\sqrt{2})|x|}, e^{-c\beta}|x|^{-4}\right)$$

This would imply that $m(\beta, z) \leqslant O(e^{-ae^{b\beta}})$, but we have no proof of this.]

The final topic of Section 4.1 is to show that the Coulomb gas in the (Mnhc) ensemble, for all $z > 0$, or in the (Mhc) ensemble, for $0 < z < 1$, has *neither short- nor long-range order*, in the sense that, for $n > 0$, $-\langle q_0 q_{ne} \rangle^{(\mathrm{hc})}(\beta, z)$ is a positive, convex function which tends to 0, as $n \to \infty$, for *arbitrary* β and ν. This is to be compared with the fact that for $\nu \geqslant 2$, $z = O(e^{a\beta})$, $\beta \gg 1$, $\langle q_0 q_x \rangle^{\mathrm{hc}}(\beta, z)$ has long-range order, in the (Mhc) ensemble.[6]

We have shown in Section 2.3, (2.32) and (2.33), that for $x \neq 0$,

$$\langle q_0 q_x \rangle^{(\mathrm{hc})}(\beta, z) = -\langle S(0) S(x) \rangle^{(\mathrm{hc})}(\beta, z) \tag{4.19}$$

with $S(x) = [d/d\phi(x)] \ln F(\phi(x))$, and

$$F(\phi(x)) = \begin{cases} \exp\left[z \cos\phi(x)\right] & \text{(Mnhc)} \\ 1 + z\cos\phi(x) & \text{(Mhc)} \end{cases}$$

Clearly $\langle S(0)^2 \rangle^{(\mathrm{hc})}(\beta, z) > 0$ [provided $z < 1$, in the (Mhc) case]. It is shown in Appendix A, (A.3) that for $x = ne$, e a lattice unit vector, $n = \mathbb{Z}\backslash\{0\}$, $0 \leqslant \langle S(0) S(x) \rangle^{(\mathrm{hc})}(\beta, z)$ is convex in x.

By (4.19), $\langle q_0 q_x \rangle^{(\mathrm{hc})}(\beta, z)$ is negative and concave in x, for $x \neq 0$, *i.e.*, *there is no short-range order.*

We pause for a short digression concerning the transfer matrix, T_q, of the Coulomb gas in the q representation (see Appendix A): in Ref. 14 it is proven that the quadratic form G with integral kernel

$$G(x - y) = (-1)^{x^1 + \cdots + x^\nu - y^1 - \cdots - y^\nu} \langle q_x q_y \rangle^{(\mathrm{hc})}(\beta, z) \tag{4.20}$$

satisfies reflection positivity (see Appendix A). A general theorem then guarantees that, for $x = ne$, e a lattice unit vector, $n \in \mathbb{Z}\backslash\{0\}$,

$$G(x) = \left\langle q_0, T_q^{n-1} q_0 \right\rangle \tag{4.21}$$

where T_q is some self-adjoint contraction on a Hilbert space with scalar product $\langle\,\cdot\,,\cdot\,\rangle$. ($T_q$ is the generalized transfer matrix.[6,14] See also Appendix A, Corollary A.3.) Now, since $\langle q_0 q_x \rangle^{(\mathrm{hc})}(\beta, z)$ is negative and concave in $x = ne$, $n \in \mathbb{Z}\backslash\{0\}$, [provided $z < 1$ in the (Mhc) case], $\operatorname{sgn} G(x) = -1)^{n-1}$, i.e., $G(x)$ is staggered. By (4.21), this implies that $T_q \ngeqslant 0$ (which is what is claimed in Appendix A; we recall that the transfer matrix, T_ϕ, of the Coulomb gas in the ϕ representation *is* nonnegative).

Next, we establish absence of long-range order in $\langle q_0 q_x\rangle^{(\mathrm{hc})}(\beta, z)$.

In Section 2.3, (2.33) we have shown that [provided $z < 1$, in the (Mhc) case]

$$0 \leqslant \left\langle |\hat{S}(k)|^2\right\rangle^{(\mathrm{hc})}(\beta, z) \leqslant \text{const} \qquad \text{for all } \beta \tag{4.22}$$

[the constant is calculated in (2.33)]. Thus $\langle|\hat{S}(k)^2|\rangle^{(\mathrm{hc})}(\beta, z)$ is bounded uniformly in k. By the Riemann–Lebesgue lemma,

$$\langle S_0 S_x\rangle^{(\mathrm{hc})}(\beta, z) \to 0 \qquad \text{as } |x| \to \infty \tag{4.23}$$

Since $\langle|\hat{q}(k)|^2\rangle^{(\mathrm{hc})}(\beta, z)$ is bounded [for $z < 1$, in the (Mhc) case], $\langle q_0 q_x\rangle^{(\mathrm{hc})}(\beta, z)$ is in $l_2(\mathbb{Z}^\nu)$, i.e., $|\langle q_0 q_x\rangle| < 0(|x|^{-\nu/2})$.

In view of (4.19) we have now proven the following theorem.

Theorem 4.4. In arbitrary dimension and for all $\beta > 0$ and $z > 0$, with $z < 1$ in the (Mhc) ensemble,

(1)

$$\left|\langle q_0 q_x\rangle^{(\mathrm{hc})}(\beta, z)\right| < O\left(|x|^{-\nu/2}\right) \qquad \text{as } |x| \to \infty$$

(absence of long-range order);

(2) for $x = ne$, e a lattice unit vector, $\langle q_0 q_x\rangle^{(\mathrm{hc})}(\beta, z)$ is negative and concave, for all $x \neq 0$ (absence of short-range order).

Remark. It is clear from the proof that Theorem 4.4 is true for arbitrary reflection positive pair potentials (not only the Coulomb potential), e.g., the Yukawa potential.

We set Theorem 4.4 in contrast to the following theorem.

Theorem 4.5. In the $(\nu \geqslant 2)$-dimensional (Mhc) ensemble with $z > e^{a\beta}$ (where a is a constant estimated in Ref. 6) and large β, $\langle q_0 q_x\rangle^{\mathrm{hc}}(\beta, z)$ has *long-range order*, and there exist at least two extremal equilibrium states $\langle\cdot\rangle^{\mathrm{hc}}_{\pm}(\beta, z)$, with

$$\langle q_x\rangle^{\mathrm{hc}}_{\pm}(\beta, z) = \pm(-1)^{x^1 + x^2 + \cdots x^\nu}$$

Thus in the (Mhc) ensemble there exists, for sufficiently large $z \approx e^{O(\beta)}$, a phase transition with order parameter, with at least two extremal equilibrium states which break translation invariance spontaneously (crystalline structure), for large β. [For some $z = z(\beta) > 1$ and large β there are in fact three extremal equilibrium states.] The proof of Theorem 4.5 can be found in Ref. 6. In Section 7 this result is extended to the dipole gases in the (Dhc) ensemble.

4.2. Characteristics of the Dipolar Phase of the Two-Dimensional Coulomb Gas

From Section 4.1 we conclude that an interesting range of parameters in the Coulomb gas about which there are only few rigorous results is $v = 2$ and (i) β large, $z > 0$ arbitrary, for the (Mnhc) ensemble; (ii) β large, $0 < z <$ const, for the (Mhc) ensemble.

This range of parameters is expected to correspond to a translation-invariant, dipolar phase of the two-dimensional Coulomb gas *without screening*. In this section we propose to characterize its properties and comment on possible methods to prove its existence.

From Section 3.2 (Remark 2) after Conjecture 3.7^{v} and Section 3.3 (Theorem 3.8) we know that the transition from the high-temperature *plasma* phase with screening to the low-temperature, *dipolar* phase without screening, *henceforth called P–D translation*, is a phase transition with order parameter. The order parameter is the fractional charge one-point correlation, $\langle e^{iq\phi(0)}\rangle(\beta, z)$, $q \in (0, 1)$. [We shall omit a superscript "hc" even when we think of the (Mhc) ensemble to which the following analysis applies, too, provided $0 < z < 1$.] We recall that, for arbitrary β,

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle(\beta, z) \geqslant \text{const}(1 + |x|)^{-q^2\beta/2\pi} \tag{4.24}$$

so that in the screening phase (β small *and* 0 b.c.)

$$\langle e^{iq\phi(0)}\rangle(\beta, z) > 0 \tag{4.25}$$

because truncated correlations cluster exponentially. The dipolar phase (β large) is characterized by

$$\langle e^{iq[\phi(0)-\phi(x)]}\rangle(\beta, z) \leqslant \text{const}(1 + |x|)^{-q^2\beta'/2\pi} \tag{4.26}$$

for some $\beta' = \beta'(\beta, q, z)$ expected to be strictly positive on $\{q\colon 0 < q < 1\}$ if β is large enough.

Thus

$$\langle e^{iq\phi(0)}\rangle(\beta, z) = 0 \tag{4.27}$$

in the dipolar phase, i.e., $\langle e^{iq\phi(0)}\rangle(\beta, z)$ is an order parameter for the $P–D$ transition. Notice that $\langle\phi(0) - \phi(x)\rangle(\beta, z) = 0$, since $\langle \cdot \rangle(\beta, z)$ is even in ϕ. Therefore if β' is independent of q, (4.26) implies

$$\left\langle\left[\phi(0) - \phi(x)\right]^2\right\rangle(\beta, z) \geqslant \text{const}\ \beta' \ln(1 + |x|) \tag{4.28}$$

[expand both sides of (4.26) to second order in q].

Next, we note that for $x = ne$, $n = 1, 2, 3, \ldots,$

$$\begin{aligned}\left\langle\left[\phi(0) - \phi(x)\right]^2\right\rangle(\beta, z) &= 2\left[\langle\phi(0)^2\rangle(\beta, z) - \langle\phi(0)\phi(x)\rangle(\beta, z)\right]\\ &\leqslant 2\langle\phi(0)^2\rangle(\beta, z)\end{aligned} \tag{4.29}$$

since $\langle\phi(0)\phi(x)\rangle(\beta, z) \geqslant 0$ by reflection positivity. Thus

$$\langle\phi(0)^2\rangle(\beta, z) \text{ is divergent} \tag{4.30}$$

in the dipolar phase.

In conclusion, (4.30) is a somewhat weaker characterization [the infinitesimal version of (4.26)] of the dipolar phase than (4.26). It obviously applies to the Villain model (Sections 3.2, 3.3) as well. [Specialists in *roughening transitions* usually prefer (4.30) over (4.26).]

It is natural to ask whether the order parameter, $\langle e^{iq\phi(0)}\rangle(\beta, z)$, can be related to the derivative of a thermodynamic quantity. To answer this question we consider the following Coulomb gas: let $d\mu_{\beta C_\Lambda}$ be the Gaussian with 0 (i.e., conducting) b.c. at $\partial\Lambda$. Consider the following partition function of a Coulomb gas with particles of charge ± 1 and activity z and particles of charge $\pm q$ and activity ζ:

$$\Xi_\Lambda(\beta, z, \zeta) = \int \prod_{x \in \Lambda} \exp\{z \cos\phi(x) + \zeta \cos[q\phi(x)]\} d\mu_{\beta C_\Lambda}(\phi) \tag{4.31}$$

The expectation in the corresponding ensemble is denoted $\langle \cdot \rangle_\Lambda(\beta, z, \zeta)$. [We only discuss particles without hard cores. The discussion applies only partially to (Mhc) ensembles.] Let

$$p_\Lambda(\beta, z, \zeta) = \frac{1}{|\Lambda|} \ln \Xi_\Lambda(\beta, z, \zeta) \tag{4.32}$$

be the finite volume pressure. The limit

$$p(\beta, z, \zeta) = \lim_{\Lambda \uparrow \mathbb{Z}^2} p_\Lambda(\beta, z, \zeta)$$

exists; see Ref. 11. Next

$$\frac{\partial p_\Lambda}{\partial \zeta}(\beta, z, \zeta) = \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \langle \cos[q\phi(x)] \rangle_\Lambda(\beta, z, \zeta) \tag{4.33}$$

The correlation inequalities of Ref. 11 (Sections 3 and 4) show that

$$\langle \cos[q\phi(x)] \rangle_\Lambda(\beta, z, \zeta) \text{ is } \textit{decreasing} \tag{4.34}$$

when Λ increases and/or ζ decreases. Thus

$$\frac{\partial p_\Lambda}{\partial \zeta}(\beta, z, \zeta) \text{ is decreasing in } \Lambda \text{ and increasing in } \zeta \tag{4.35}$$

Using (4.32)–(4.35) and convexity of $p_\Lambda(\beta, z, \zeta)$ in ζ one easily shows that

$$\frac{\partial p}{\partial \zeta}(\beta, z, \zeta \pm) = \lim_{\Lambda \uparrow \mathbb{Z}^2} \frac{\partial p_\Lambda}{\partial \zeta}(\beta, z, \zeta \pm) = \langle \cos[q\phi(0)] \rangle(\beta, z, \zeta \pm),$$

where $\langle \cdot \rangle(\beta, z, \zeta)$ is the thermodynamic limit of $\langle \cdot \rangle_\Lambda(\beta, z, \zeta)$ and $F(\zeta \pm)$

$= \lim_{\epsilon\downarrow 0} F(\zeta \pm \epsilon)$. By (4.34),

$$\langle \cos[q\phi(0)] \rangle(\beta, z, \zeta +) = \langle \cos[q\phi(0)] \rangle(\beta, z, \zeta), \tag{4.36}$$

as the two limits, $\zeta' \searrow \zeta$ and $\Lambda \nearrow \mathbb{Z}^2$, can be interchanged by monotonicity. Thus $(\partial p/\partial \zeta)(\beta, z, 0+) = \langle \cos[q\phi(0)] \rangle(\beta, z, 0)$, and since $\langle \cdot \rangle(\beta, z, 0) = \langle \cdot \rangle(\beta, z)$ is even in ϕ,

$$\frac{\partial p}{\partial \zeta}(\beta, z, 0+) = \langle e^{iq\phi(0)} \rangle(\beta, z) \tag{4.37}$$

which is the desired relation.

Therefore the P–D transition can also be characterized by

$$\begin{aligned} &\frac{\partial p}{\partial \zeta}(\beta, z, 0+) > 0 \quad \text{in the } P \text{ (screening) phase} \\ &\frac{\partial p}{\partial \zeta}(\beta, z, 0+) = 0 \quad \text{in the conjectured } D \text{ phase} \end{aligned} \tag{4.38}$$

We now sketch an argument suggesting that (4.27) holds for large β. To this purpose, it is useful to compare p with the pressure p^F given by (4.31) and (4.32) but where $d\mu_{\beta C_\Lambda} = d\mu_{\beta C}$ has free b.c. We then apply Lemma 4.2 to the right-hand side of (4.31) to conclude that $\Xi_\Lambda^F(\beta, z, \zeta)$ and hence $p_\Lambda^F(\beta, z, \zeta)$ are functions of z^2 and ζ^2, because neutrality of a charge configuration requires an even number of particles with charge $\pm$ and charge $\pm q$, as $q \in (0, 1)$. Thus $p^F(\beta, z, \zeta) = \lim_{\Lambda\uparrow\mathbb{Z}^2} p_\Lambda^F(\beta, z, \zeta)$ is *even* in z and in ζ.

For β large enough ($\gtrsim 8\pi q^{-2}$), low-order terms in an expansion of $p^{(F)}(\beta, z, \zeta)$ in z and ζ about $z = \zeta = 0$ are *convergent* (i.e., infrared finite in the thermodynamic limit) and independent of b.c., i.e., the same for 0 and free b.c. Since p^F is even in z and ζ, the coefficients of z^{2n+1}, ζ^{2m+1} in this expansion all vanish. Presumably, the expansion in z and ζ about $z = \zeta = 0$ is divergent, but it is reasonable to expect it is asymptotic.

Thus, we conjecture that for β large enough (1) $p^F(\beta, z, \zeta)$ is continuously differentiable in ζ at $\zeta = 0$, for $0 < z$ small enough; by evenness that would imply $(\partial p^F/\partial \zeta)(\beta, z, 0) = 0$, for small z; and (2) for sufficiently small z and ζ, $p(\beta, z, \zeta) = p^F(\beta, z, \zeta)$.

By (4.25), (4.37), and (4.38), a proof of (1) and (2) above would also imply the existence of a P–D transition.

Next we give a heuristic argument suggesting that

$$m(\beta, z) \searrow 0 \qquad \text{as } \beta \nearrow \beta_c \qquad \text{with } \beta_c \approx 8\pi \tag{4.39}$$

In Ref. 10, the continuum limit of the two-dimensional (Mnhc) Coulomb gas has been constructed for all $\beta < 4\pi$. By using the scaling properties of the two-dimensional Coulomb potential one can show that the inverse correlation length (mass) has the form

$$m(\beta,z)^2 = \mu(\beta)z^{2/(2-\beta/4\pi)} \tag{4.40}$$

for some function $\mu(\beta) \geq 0$; see Ref. 10.

Perturbative arguments[40] suggest that, after a divergent, additive renormalization of the pressure, the continuum limit of the Coulomb gas exists, and Eq. (4.40) remains true, for all $\beta < 8\pi$. Now suppose that $\mu(\beta)$ has at most a power law divergence at $\beta = 8\pi$. Then

$$\lim_{\beta \uparrow 8\pi} m(\beta,z) = 0 \qquad \text{for } z < 1 \tag{4.41}$$

This suggests that, on the lattice, the critical point, β_c, of the $P-D$ transition is $\approx 8\pi$. The point $\beta = 8\pi$ also seems to be a critical point in a recent, exact study of the sine-Gordon theory which is isomorphic to the continuum Coulomb gas by Sklyamin, Takhtadzhyan, and Faddeev.[41]

In Section 5 we develop techniques and estimates which we hope are suitable to rigorously establish the existence of a $P-D$ transition for the two-dimensional lattice Coulomb gas in the (Mhc) ensemble and prove inequality (4.26) for large β and $0 < z < 1$. In Section 5 we study a two-dimensional dipole gas in a (Dhc) or (Dnhc) ensemble for which we prove (4.26) for arbitrary β. The techniques of Section 5 suffice to analyze gases of general, neutral multipoles, provided the activity of multipoles of large size is suitably small.

We therefore propose to approximate the two-dimensional Coulomb gas by gases of neutral multipoles of arbitrary sizes in a convergent fashion and such that inequality (4.26) remains true in the limit. (Notice that Lemma 4.2 says that in principle such an approximation is possible, at least for free b.c.)

5. THE DECAY OF THE CHARGE-CHARGE CORRELATION IN DIPOLE GASES

In this section we study the charge–charge correlation in a sea of dipoles. We shall concentrate on two specific ensembles. Let $d\mu_\beta(\phi)$ denote the Gaussian measure of covariance $\beta(-\Delta+\epsilon^2)^{-1}$, in the limit $\epsilon \to 0$. Here Δ denotes the finite difference Laplacean. The first ensemble describes dipoles with no hard core (Dnhc):

$$\langle \cdot \rangle(\beta,z) = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} \Xi_\Lambda(\beta,z)^{-1} \cdot \int - e^{zU(\Lambda)} d\mu_\beta(\phi) \tag{5.1}$$

where

$$U(\Lambda) = \sum_{|j-j'|=1} \cos\alpha\left[\phi(j) - \phi(j')\right], \qquad \alpha = 1 \text{ or } 2\pi$$

and Ξ_Λ is the partition function. By scaling the constant α can always be chosen to be 1, but for notational convenience we will choose $\alpha = 2\pi$ at one place.

The expectation for dipoles with hard core (Dhc) is given by "$\lim \Lambda \uparrow \mathbb{Z}^\nu$" of

$$\langle \cdot \rangle_\Lambda(\beta, z) = \Xi_\Lambda^{-1} \int - \prod_{j \in \mathsf{L}} \left\{ 1 + z \sum_e \cos \alpha \big[\phi(j) - \phi(j+e) \big] \right\} d\mu_\beta(\phi) \quad (5.2)$$

where e ranges over unit lattice vectors and $\mathsf{L} \overset{\text{e.g.}}{=} 4\mathbb{Z}^\nu \cap \Lambda$.

The above ensembles are very special cases of those discussed in Section 2. Although we shall prove our results in detail only for the above expectations, nearly all our results extend to the more general class of dipole gases discussed in Section 2. In fact, at the end of this section we shall analyze dipoles of arbitrary length L and prove the analog of the following result.

Theorem 5.1. Let $\alpha = 1$. For all β, z, in the (Dnhc) ensemble, we have

$$\langle e^{i[\phi(y) - \phi(x)]} \rangle (\beta, z) \leqslant C_{\epsilon,\beta} e^{-g_1 \log|y-x|}$$

where

$$g_1 = \frac{\beta}{2\pi} \left[1 + \beta(z + 2\epsilon) \right]^{-1}$$

and ϵ can be chosen arbitrarily small. Also, for the (Dhc) ensemble, if $|z| \leqslant \frac{1}{16} e^{\beta/8}$, we have

$$\langle e^{i[\phi(y) - \phi(x)]} \rangle (\beta, z) \leqslant C_\beta e^{-g_2 \log|y-x|}$$

with

$$g_2 = \frac{\beta}{2\pi} \left(1 + 4\beta z e^{-\beta/8} \right)^{-1}$$

As explained in Section 3, the two-point correlation of the Villain model is, by duality, equal to

$$\langle e^{i(\theta_0 - \theta_x)} \rangle^v (\beta) = \langle A_x \rangle^{\hat{v}} (\beta)$$

where

$$A_x = \exp\left[\frac{1}{\beta} \phi(\partial_2 f^x) - \frac{|x|}{2\beta} \right] \quad (5.3)$$

and

$$f^x(j) = \begin{cases} 1, & 0 \leqslant j_1 \leqslant x \quad \text{and} \quad j_2 = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.4)$$

As a first step toward understanding the decay properties of (5.3) we shall bound $\langle A_x\rangle(\beta,z)$ from below in the dipole ensembles (5.1) and (5.2), with $\alpha = 2\pi$.

Theorem 5.2. Let $\alpha = 2\pi$. For all β, z in the (Dnhc) ensemble (5.1) we have

$$\langle A_x\rangle(\beta,z) \geqslant e^{-g\log|x|} \tag{5.5}$$
$$g = \beta^{-1}/2\pi + z\cdot\text{const}$$

For the (Dhc) ensemble (5.2), if $|z| \leqslant \frac{1}{4}e^{+\beta/8}$, then (5.5) holds with

$$g = \beta^{-1}/2\pi + \text{const}\, ze^{-\beta/8}$$

Remark. To prove Theorem 5.1 we shall use the method of complex translations,[29] $\phi(j)\to\phi(j) + ia(j)$, for suitable a, whereas in the proof of Theorem 5.2, we shall apply a *real translation*, $\phi(j)\to\phi(j) + a(j)$.

Proof of Theorem 5.1. For notational simplicity we first set $y = 0$. We apply a complex translation of the field $\phi^{(29)}$:

$$\phi(j)\to\phi(j) + i\gamma a(j) \tag{5.6}$$

where γ depends on the ensemble and is chosen later, and

$$a(j) = C(j,0) - C(j,x) \tag{5.7}$$

with C the kernel of $-\Delta^{-1}$. The function $a(j)$ satisfies the following relations:

$$\sum_{|j-j'|=1} \left[a(j) - a(j')\right]^2 = \langle a, -\Delta a\rangle$$
$$= a(0) - a(x) \approx (\log|x|)/\pi \tag{5.8}$$

for large $|x|$, and for $|j - j'| = 1$

$$\begin{aligned} |a(j) - a(j')| &\leqslant \text{const}\left(\frac{1}{|j|+1} + \frac{1}{|j-x+1|}\right) \\ |a(j) - a(j')| &\leqslant \text{const}|x|/|j|^2 \end{aligned} \tag{5.9}$$

Let us first consider the dipole gas in the (Dnhc) ensemble. In this case we choose

$$\gamma = (\beta^{-1} + z)^{-1} = \beta(1 + \beta z)^{-1}$$

Under the change of variables (5.6) the exponent of the functional measure becomes

$$-(1/2\beta)\sum_j\left[\nabla(\phi+i\gamma a)(j)\right]^2+z\sum_{|j-j'|=1}\cos\{\phi(j)-\phi(j')+i\gamma\times\left[a(j)-a(j')\right]\}-\gamma\left[a(0)-a(x)\right]+i\left[\phi(0)-\phi(x)\right]$$

We estimate the integrand in the functional integral by taking absolute values, i.e., the real part of the exponent, and using (5.8) and (5.9). The real part of the exponent is

$$-(1/2\beta)\sum_j(\nabla\phi)^2(j)+z\sum_{|j-j'|=1}\cos\left[\phi(j)-\phi(j')\right]\cosh\gamma\left[a(j)-a(j')\right]+\frac{\gamma^2}{2\beta}\left[a(0)-a(x)\right]-\gamma\left[a(0)-a(x)\right]$$

For $|j|\gg\gamma, |j-x|\gg\gamma$, we have by (5.9)

$$\cosh\{\gamma\left[a(j)-a(j')\right]\}-1\leqslant(1+\epsilon)\gamma^2\left[a(j)-a(j')\right]^2/2$$

[For $|j|, |j-x|\leqslant O(\gamma)$, we simply estimate $\cosh(\cdot)-1$ by a constant.] Thus by (5.8) the exponent is bounded by

$$-(1/2\beta)\sum_j(\nabla\phi)^2(j)+z\sum_{|j-j'|=1}\cos\left[\phi(j)-\phi(j')\right]-\gamma/2(1-\epsilon\gamma)\left[a(0)-a(x)\right]+\text{Const}$$

with $\gamma=\beta(1+\beta z)^{-1}$,

$$\gamma/2(1-\epsilon\gamma)\geqslant\beta/2(1+\beta z+z\epsilon\beta)^{-1}$$

for some small ϵ. Integration over ϕ now cancels the partition function, and we have

$$\left\langle e^{i[\phi(0)-\phi(x)]}\right\rangle\leqslant e^{-\beta/2\pi(1+\beta z+2\epsilon\beta)^{-1}\log|x|}$$

This completes the proof of Theorem 5.1 for the (Dnhc) ensemble.

Next we turn to the case of dipoles with a hard core, i.e., (Dhc). In this case, we first prove a lemma which enables us to take advantage of the small *effective* fucacity of dipoles.

Lemma 5.3. Let $F\{\phi\}$ be a function of $\{\phi(j)\}$ with $j\neq j_0$, and set

$$\bar{z}=ze^{-\beta/8},\qquad\bar{\phi}(j_0)=\tfrac{1}{4}\sum_{|j-j_0|=1}\phi(j)\tag{5.10}$$

Then

$$\int\left[1+z\cos(\phi(j_0)-\phi(j_1))\right]F\,d\mu_\beta(\phi)=\int\left[1+\bar{z}\cos\left(\bar{\phi}(j_0)-\phi(j_1)\right)\right]F\,d\mu_\beta(\phi)$$

Remarks. Lemma 5.3 is simply an explicit computation of the conditional expectation of $\cos[\phi(j_0) - \phi(j_1)]$, given $\{\phi(j)\}\ j \neq j_0$. The simplicity of the result is due to the Markov and Gaussian property of $d\mu_\beta(\phi)$.

Because of the nonoverlapping condition on the dipoles we have the identity

$$\int \prod_{j \in L} \left\{ 1 + z \sum_e \cos\left[\phi(j) - \phi(j+e)\right] \right\} e^{i[\phi(y) - \phi(x)]} d\mu_\beta(\phi)$$

$$= \int \prod_{j \in L} \left\{ 1 + \bar{z} \sum_e \cos\left[\bar{\phi}(j) - \phi(j+e)\right] \right\} e^{i(\phi(y) - \phi(x))} d\mu_\beta(\phi) \quad (5.11)$$

whenever x and y do *not* belong to L.

Proof. The lemma follows by an explicit integration of the $\phi(j_0)$ variable. This is easy, because the integral is Gaussian:

$$\int e^{i\phi(j_0)} \prod_{|j - j_0| = 1} e^{-[\phi(j_0) - \phi(j)]^2/2\beta} d\phi(j_0)$$

$$= \int \prod_{|j - j_0| = 1} e^{-[\phi(j_0) - \phi(j)]^2/2\beta} d\phi(j_0) e^{-\beta/8} e^{i\bar{\phi}(j_0)}$$

where we have used

$$\int \exp\left\{ -2\phi(j_0)^2/\beta + \phi(j_0)\left[i + 8\,\overline{\phi(j_0)}\,/\beta \right] \right\} d\phi(j_0)$$

$$= \int \exp\left[-2\phi(j_0)^2/\beta \right] d\phi(j_0) e^{-\beta/8} e^{i\bar{\phi}(j_0)} e^{8\,\overline{\phi(j_0)}^2/\beta} \quad \blacksquare$$

Proof of Theorem 5.1. The proof for hard core dipoles also follows by complex translations and taking absolute values. We first apply Lemma 5.3 to reexpress the numerator and partition function as in (5.11). (We assume that y is near 0 but x and y do not belong to L. The general case will be discussed at the end of this section.) Let $a(j)$ be as in (5.7). The inequality

$$|r + ib| \leqslant r + b^2/2r \leqslant r e^{b^2/2r}, \qquad r > 0$$

can be applied to show that for $|\bar{z}| \leqslant 1/16$ we have

$$\left| 1 + \bar{z} \sum_e \cos\left[\overline{\delta_e \phi}(j) + i\gamma \overline{\delta_i} a(j) \right] \right|$$

$$\leqslant \left\{ 1 + \bar{z} \sum_e \cos\left[\overline{\delta_e \phi}(j) \right] \cosh\left[\gamma \overline{\delta_e a}(j) \right] \right\} \cdot e^{\gamma \bar{z} \delta_e a(j)^2}$$

$$\leqslant \left\{ 1 + \bar{z} \sum_e \cos\left[\overline{\delta_e \phi}(j) \right] \right\} e^{2[\gamma \bar{z} \delta_e a(j)]^2}$$

provided $|j - x| \gg 1$ and $|j - y| \gg 1$. Here we have used the notation

$$\overline{\delta_e \phi}(j) = \bar{\phi}(j) - \phi(j + e)$$

and

$$\delta_e \phi(j) = \phi(j) - \phi(j+e)$$

Note that since

$$\overline{a(j)} - a(j) = \tfrac{1}{4}(\Delta a)(j) = 0, \qquad j \neq x, y$$

we have

$$\overline{\delta_e a}(j) = \overline{a(j)} - a(j+e) = a(j) - a(j+e)$$

when $j \neq x, y$.

The Gaussian contribution, after taking absolute values, equals $C \exp(\gamma^2/2\beta)[a(x) - a(y)]$, as in the proof for the (Dnhc) ensemble. The sum $2(\gamma\bar{z})2\sum_{j,e}[\delta_e a(j)]^2$ is clearly bounded by

$$2(\gamma\bar{z})^2[a(x) - a(y)]$$

The ϕ integration of the numerator exactly cancels the denominator.

Collecting all coefficients of $a(x) - a(y)$ we see that $\gamma = \beta(1 + 2\beta\bar{z})^{-1}$ is the optimal choice. As in the proof for the (Dnhc) ensemble we finally obtain

$$\left\langle e^{i[\phi(x) - \phi(y)]} \right\rangle \leqslant C_\gamma e^{-(\beta/2\pi)(1+4\beta\bar{z})^{-1} \log|x-y|}$$

which completes the proof of Theorem 5.1. ■

Remarks. We have seen that the decay of the charge–charge correlation in the dipole gas has a fairly elementary proof using the ϕ representation. If one attempts to obtain such results directly in the q (or gas) representation the required estimates appear to be far more complicated. The q representation does have the advantage of making the small activity of the dipoles manifest. What Lemma 3 does is to give us a kind of mixed q–ϕ representation; for had we integrated all the ϕ variables we would have precisely obtained the q (or gas) representation. Thus our approach amounts to a *phase space* analysis in function space.

Next we prove Theorem 5.2. We shall consider only the (Dhc) case, since the other case is easier. We make the real change of variables,

$$\phi(j) \to \phi(j) + a(j)$$

with $a(j)$ given by

$$a(j) = \sum C(j,k)\partial_2 f^x(k) \tag{5.12}$$

Under this transformation, the linear terms cancel, and by Lemma 5.3 the exponent of the interaction becomes

$$\sum_{j \in L} \log\left\{1 + \bar{z}\sum_e \cos 2\pi\left[\overline{\delta_e\phi}(j) + \overline{\delta_e a}(j)\right]\right\} - F(x)$$

where

$$F(x) = \frac{1}{2\beta}\left[\sum_j (\nabla a)^2(j) + |x| - 2\sum_j a(j)(\partial_2 f^x)(j)\right]$$
$$= \frac{1}{2\beta}\left[|x| + \sum_j (\partial_2 a)(j) f^x(j)\right]$$

Note that $(\partial_1 f^x)(j) = \delta(j,0) - \delta(j,x)$, and

$$\begin{aligned}(\partial_2 a)(j) &= \sum_k \partial_2 C(j-k)(\partial_2 f^x)(k)\\ &= \sum_k (\partial_2^2 C)(j-k) f^x(k)\\ &= \sum_k (\Delta C)(j-k) f^x(k)\\ &\quad - \sum_k (\partial_1^2 C)(j-k) f^x(k)\\ &= -f^x(j) - \sum_k (\partial_1 C)(j-k)(\partial_1 f^x)(k)\\ &= -f^x(j) - \partial_1 C(j) + \partial_1 C(j-x) \end{aligned} \tag{5.13}$$

Summing $\partial_2 a(j)$ over j we see that $|x|$ is canceled and thus

$$F(x) \approx \frac{\log|x|}{2\pi\beta}$$

Also we have

$$\begin{aligned}(\partial_1 a)(j) &= \sum \partial_1 C(j-k)\partial_2 f^x(k)\\ &= \sum \partial_2 C(j-k)(\partial_1 f^x)(k)\\ &= \partial_2 C(j) - \partial_2 C(j-x)\end{aligned}$$

Since a is harmonic (mod 1), $\overline{\delta a}(j) = \delta a(j)$, (mod 1). From (5.13) and the above equation we see that $\delta a(j)$ satisfies (mod 1) the estimates (5.8) and (5.9). We now introduce the functions $D_j(\phi)$

$$\begin{aligned}D_j &= \log\Big\{1 + \bar{z}\sum_e \cos 2\pi\big[\,\overline{\delta_e\phi}\,(j) + \delta_e a(j)\big]\Big\}\\ &\quad - \log\Big\{1 + \bar{z}\sum_e \cos\big[2\pi\,\overline{\delta_e\phi}\,(j)\big]\Big\}\\ &= \bar{z}\sum_e \Big\{\cos\big[2\pi\,\overline{\delta_e\phi}\,(j)\big]\big(\cos\big[2\pi\delta_e a(j)\big] - 1\big)\\ &\quad + \sin\big[2\pi\delta_e\phi(j)\big]\big(\sin\big[2\pi\delta_e a(j)\big]\big)\Big\} + \bar{z}^2 O\big[\delta a(j)^2\big]\end{aligned}$$

In the last equality we have used the double-angle formula and the Taylor series for $\log(1+x)$. Now set $D = \sum D_j$, and then we have using Jensen's inequality and the fact that $\sin[2\pi\delta_e\phi(j)]$ is odd

$$\begin{aligned}\langle A_x\rangle(\beta,z) &= \langle e^{-D}\rangle(\beta,z)e^{-F(x)}\\ &\geqslant \exp\left\{-\bar{z}\,\mathrm{const}\sum_{j,e}\left[\delta_e a(j)\right]_1^2-(1/2\pi\beta)\log|x|\right\}\\ &\geqslant \exp\left[-\left(\frac{1}{2\pi\beta}+\bar{z}\,\mathrm{const}\right)\log|x|\right]\end{aligned}$$

with $\{\delta_e a(j)\}_1 = \delta_e a(j)$, mod 1. ■

Next we turn to the more general case of dipoles of different lengths. Suppose we consider the fractional charge correlation in a two-dimensional Coulomb gas. Oppositely charged particles will tend to form dipoles of various lengths with dipoles of length L having a small effective fugacity $\approx \exp[-(\beta/2\pi)\log L]$, but with an entropy proportional to L^4. To mimic this situation we consider two lattices

$$\mathsf{L}_1 = d\mathbb{Z}^2\cap\Lambda \quad\text{and}\quad \mathsf{L}_2 = \left[dL\mathbb{Z}^2+\left(\frac{d}{2},0\right)\right]\cap\Lambda$$

Let

$$\delta_L\phi(j) = \phi(j)-\phi(j+Le)$$

with $e = e_j$ a unit lattice vector, and define

$$\begin{aligned}\langle e^{i[\phi(x)-\phi(y)]/2}\rangle_\Lambda = \Xi_\Lambda^{-1}\int \prod_{j\in\mathsf{L}_1}\left[1+z_1\cos\delta_1\phi(j)\right]\prod_{j\in\mathsf{L}_2}\left[1+z_L\cos\delta_L\phi(j)\right]&\\ \times e^{i[\phi(x)-\phi(y)]/2}d\mu_\beta(\phi)&\qquad(5.14)\end{aligned}$$

It is convenient to let $\bar{\mathsf{L}}_1$ and $\bar{\mathsf{L}}_2$ denote the collection of squares $B_1(j)$, $B_2(j)$ centered at the sites of L_1 and L_2 having sides of length d and dL, respectively. Note that $\mathsf{L}_1\cap\mathsf{L}_2=\emptyset$ so that the positions of the dipoles do not overlap.

The choice $z_1\approx d^2$ and $z_L\approx L^4d^2$ mimics the entropy since there are approximately d^2, L^4d^2 dipoles associated to each site of L_1, L_2, respectively. Now we want to replace (5.14) by a similar expression but with z_1, z_L replaced by effective (renormalized) activities,

$$0\leqslant \bar{z}_1\leqslant z_1e^{-\beta/8},\qquad 0\leqslant \bar{z}_L\leqslant z_L\,\mathrm{const}\exp\left[-\left(\beta-\frac{\mathrm{const}}{d^2}\right)\frac{\log L}{2\pi}\right] \qquad (5.15)$$

This result will be obtained by integrating large blocks of ϕ's using complex translations. Consider the dipole corresponding to $\delta_L\phi(j)$. For each $j\in\mathsf{L}_2$, let $\zeta_j(k)$ be a function on $\mathbb{Z}^2$ satisfying

$$\zeta_j(k)=1,\qquad |j-k|\leqslant 3L/2,\qquad \zeta_j(k)=0,\qquad |j-k|\geqslant 2L \quad (5.16)$$

and

$$|\nabla \zeta_j(k)| \leqslant \text{const}/L, \qquad 0 \leqslant \zeta_j(k) \leqslant 1 \tag{5.17}$$

Define

$$\begin{aligned} b_j(k) &= \zeta_j(k)c(k), \qquad k \notin \bar{\mathrm{L}}_1 \\ b_j(k) &= \zeta_j(i)c(i), \qquad |k-i| \leqslant 2, \qquad k \in \mathrm{L}_1 \end{aligned} \tag{5.18}$$

where

$$c(k) = C(j,k) - C(j+Le,k)$$

Finally we introduce

$$\begin{aligned} \overline{\delta_L \phi}\,(j) &= \sum \nabla b_j(k) \cdot \nabla \phi(k) - \phi(j) + \phi(j+Le) \\ \overline{\delta_1 \phi}\,(i) &= \bar{\phi}(i) - \phi(i+e) \end{aligned} \tag{5.19}$$

Lemma 5.4. Suppose the observables do not overlap with the dipoles of the ensemble. The expectation (5.14) remains unchanged if we replace each z and $\delta\phi(j)$, by $\bar{z}$, $\overline{\delta\phi}(j)$ which satisfy (5.15) and (5.19).

Proof. We first apply Lemma 5.3 to obtain $\overline{\delta_1\phi}$ and $\bar{z}_1$. To rewrite $\cos[\delta_L\phi(0)]$ we make the change of variables

$$\phi(k) \to \phi(k) + i\beta b_0(k) \tag{5.20}$$

which leaves $\overline{\delta_1\phi}$ unaffected, because by (5.18) $\overline{\delta_1 b}(j) = 0$. The function ζ ensures that functions localized outside $\{|k| \leqslant 2L\}$ are unchanged. Thus we need to see how (5.20) affects $\exp[i\delta_L\phi(0)]$ and $d\mu_\beta(\phi)$:

$$\begin{aligned} &\exp\Big\{ -\sum \big[\nabla(\phi + i\beta b_0(j)) \big]^2(j)/2\beta + i\delta_L\phi(0) - \beta(\delta_L b_0(0)) \Big\} \\ &\quad = \exp\Big[-\sum (\nabla\phi)(j)^2/2\beta + i\,\overline{\delta_L\phi}\,(0) - \beta(\delta_L b_0(0)) + \sum \beta(\nabla b_0)^2(j)/2 \Big] \end{aligned}$$

It is easy to see using (5.16), (5.17), and (5.9) that with $\zeta = \zeta_0$

$$\begin{aligned} \frac{\beta}{2} \sum_k \big[\nabla(\zeta c)\big]^2(k) &= \frac{\beta}{2} \sum_k \big[(\nabla c)\zeta + (\nabla\zeta)c\big]^2(k) \\ &\leqslant \frac{\beta}{2} \sum (\nabla c)^2(k)\zeta(k) + \text{const} \\ &\leqslant \frac{\beta}{2\pi} \log L + \text{const} \end{aligned}$$

Again by (5.9) we have

$$\sum_k \left\{ (\nabla b_0)^2(k) - [\nabla(\zeta c)]^2(k) \right\} = \sum_{k \in \mathsf{L}_1} [\nabla(b_0 - \zeta c)]^2(k)$$

$$\leqslant \log L \frac{\text{const}}{d^2}$$

The same argument applies to renormalize $\cos[\delta_L \phi(j)]$. $j \in \mathsf{L}_2, j \neq 0$. ■

Remark. If $d \geqslant 4$ is not large it is advantageous to replace (5.13) by $\phi(k) \to \phi(k) + i\gamma b(k)$ with γ small. One can then obtain

$$|\bar{z}_1| \leqslant z_L \exp - (\gamma\beta \log L)$$

without having to choose d very large. For an alternate method see Appendix B.

Theorem 5.5. If x, y do not overlap with dipole positions, and β is so large that $|\bar{z}_1| \leqslant 1/4$, $|\bar{z}_2| \leqslant 1/4$ then

$$\left\langle e^{i[\phi(x) - \phi(y)]/2} \right\rangle \leqslant C_\beta e^{-g \log|x-y|} \tag{5.21}$$

where

$$g = \frac{\beta}{8\pi}\left(1 + \beta \frac{\overline{z_1} + \overline{z_2}}{d^2} \text{const}\right)^{-1}$$

with const independent of β, L, d and $\langle \cdot \rangle$ is given by (5.14) in the limit $\Lambda = \mathbb{Z}^2$.

Proof. By Lemma 5.4 the substitution $z \to \bar{z}$, $\delta\phi \to \overline{\delta\phi}$ applied to the numerator and partition function does not change the expectation. Let $a(j) = c(j,x) - c(j,y)$ and translate

$$\phi(j) \to \phi(j) + i\frac{\gamma}{2} a(j)$$

From (5.19) we have

$$\overline{\delta_L[\phi + i(\gamma/2)a]}(0) = \overline{\delta_L \phi}(0) + i\gamma[a(0) - a(L)]/2$$

because $\Delta a(j) = \delta(x - j) - \delta(y - j)$. By (5.9), for j restricted to $|j - x| \leqslant 2|x - y|$,

$$\gamma^2 \sum_{j \in \mathsf{L}_2} |\delta_L a(j)|^2 \leqslant \gamma^2 L^2 \sum_{j \in \mathsf{L}_2} \text{const}\left(\frac{1}{|j-x|+1} + \frac{1}{|j-y|+1}\right)^2$$

$$\leqslant \gamma^2 \frac{\text{const} \log|x-y|}{\pi d^2}$$

For $|j-x| \geqslant 2|x-y|$ the sum is bounded by a constant, see (5.9). Thus the total factor arising from the translation is bounded by

$$\left\{ \frac{\gamma}{4} - \left[\frac{\gamma^2}{8\pi} + \gamma^2 \left(\frac{\bar{z}_1 + \bar{z}_2}{d^2} \right) \text{const} \right] \right\} \frac{\log|x-y|}{\pi}$$

The optimal choice of γ yields (5.21). ■

Remarks. In the proof of Theorems 5.1 and 5.4 we have used the fact that x, y do not overlap with the dipoles of the interaction. This requirement is unnecessary, as we now show. For Theorem 5.1 suppose $y = 0$ and $x \notin 4\mathbb{Z}$. Consider the factor

$$e^{i[\phi(0)-\phi(x)]} \Big\{ 1 + z \sum_e \cos[\phi(0) - \phi(e)] \Big\}$$
$$= e^{i[\phi(0)-\phi(x)]} + \frac{z}{2} \sum_e e^{i[2\phi(0)-\phi(e)-\phi(x)]} + \frac{z}{2} \sum_e e^{i[\phi(e)-\phi(x)]}$$

If we estimate each term as before, treating it as nonoverlapping observable, we get the bound

$$\Xi^{-1}/\Xi'(1+2z)e^{-g_2 \log|x-y|}$$

The partition function Ξ' has the factor

$$1 + z \sum_e \cos[\phi(0) - \phi(e)]$$

deleted hence, by a simple argument, $\Xi'/\Xi \leqslant 1$. A similar argument applies to Theorem 5.4 but here we may need to use the *fractional* charge of the observable so that the observable and interaction dipoles do not cancel.

It is easy to extend the above results to systems of the form

$$\prod_{j \in L_1} \Big\{ 1 + z_1 \sum_{k,k' \in B_1(j)} \cos[\phi(k) - \phi(k')] \Big\}$$
$$\times \prod_{j \in L_2} \Big\{ 1 + z_2 \sum_{\substack{k,k' \in B_2(j) \\ L/2 \leqslant |k-k'| \leqslant L}} \cos[\phi(k) - \phi(k')] \Big\}$$

We can apply the renormalization of fugacity principle as before. Note that there are $\approx L^4$ terms in the sum over k, k' in B_2; thus, since n!e obtain from the renormalization of the activity a factor of $e^{\log L/2\pi}$ we need to require $\beta > 8\pi$, so that

$$z_2 L^4 e^{\frac{\beta}{2\pi} \log L} < 1 \qquad \text{for large} L$$

and our measure is positive, whence our techniques apply.

We hope that the techniques developed in this section combined with an expansion of the two-dimensional Coulomb gas in terms of neutral multipole configurations of arbitrary size and (unfortunately very tedious) combinatorial estimates will permit one to prove the existence of a P–D transition in the (Mhc) ensemble, before long. For an alternate treatment of the renormalization of dipole activities, using purely electrostatic techniques, see Appendix B.

6. THE MERMIN ARGUMENT

By establishing a generalized Mermin theorem we shall prove a *lower* bound in momentum space on the ϕ two-point function, where the expectation is given by (5.1) or (5.2). It is convenient to replace $\beta(-\Delta)^{-1}$ by $\beta(-\Delta_\Lambda+\epsilon)^{-1}$. The subscript Λ indicates *periodic* boundary conditions at $\partial\Lambda$, and ϵ is an infrared regulator to be removed after the thermodynamic limit has been taken. Let

$$\hat{\phi}(p) \equiv |\Lambda|^{-1/2} \sum_{j\in\Lambda} \phi(j) e^{ij\cdot p}$$

and let $\Delta(p)$ be the Fourier transform of $-\Delta$.

Theorem 6.1. Let z, β be as in Theorem 5.1. Then for the (Dnhc) ensemble we have

$$\left\langle |\hat{\phi}(p)|^2 \right\rangle \geqslant (\beta^{-1}+z)^{-1}\Delta(p)^{-1} \tag{6.1}$$

For the (Dhc) ensemble

$$\left\langle |\hat{\phi}(p)|^2 \right\rangle \geqslant (\beta^{-1}+\bar{z})^{-1}\Delta(p)^{-1} \tag{6.1$'$}$$

where $\bar{z} \leqslant \text{const}\, ze^{-\beta/8}/(1-ze^{-\beta/8})^2$.

As a corollary to these inequalities we shall show that the correlation of two infinitesimal test dipoles immersed in a (Dnhc) or (Dhc) dipole gas does not decay integrably fast.

Before formulating our generalized Mermin–Wagner theorem we illustrate how the methods of the previous section enable us to establish (6.1). In fact the standard Mermin theorem is an infinitesimal form of the results in Section 5. (Similarly, the infrared bounds are an infinitesimal form of Gaussian domination; see Refs. 20 and 14.) Let f be a function on the lattice and set $\phi(f) = \sum_j \phi(j) f(j)$. By subtracting 1 from both sides of the inequality

$$\left\langle e^{i\epsilon\phi(f)} \right\rangle \leqslant e^{-\epsilon^2\beta'\langle f, Cf\rangle} \tag{6.2}$$

and keeping the second-order terms in ϵ we obtain

$$\beta'\langle f, Cf\rangle \leqslant \langle \phi(f)^2 \rangle \tag{6.1$''$}$$

hence (6.1) follows by letting $f(j) = |\Lambda|^{-1/2} e^{ip \cdot j}$. Inequality (6.2) follows for dipole systems as in Section 5, using the translation $\phi(j) \to \phi(j) + i\epsilon\beta' a_j$, where

$$a_j = \sum_{j'} C(j, j') f(j') \tag{6.3}$$

The estimates are nearly identical to the ones in Section 5 if we use the relation

$$\sum_{|j-j'|=1} (a_j - a_{j'})^2 = -\langle a, \Delta a \rangle = \langle f, Cf \rangle$$

Note that this technique (as well as the one that follows) works in arbitrary dimension.

To set up the Mermin argument (infinitesimal form), let $H(\phi)$ be a real function of $\phi(j)$, j belonging to a box Λ, and define Z_Λ so that

$$\langle \cdot \rangle = Z_\Lambda^{-1} \int \cdot \, e^{-H(\phi)} \prod_{j \in \Lambda} d\phi(j)$$

is a probability measure. Let D be a first-order differential operator on $L^2(\mathbb{R}^{|\Lambda|})$.

Lemma 6.2. For regular functions F, H we have

$$|\langle [D, F] \rangle| \leqslant \langle [D, [\overline{D, H}\,]] \rangle^{1/2} \langle |F|^2 \rangle^{1/2}$$

Note, all commutators are functions, because D is first order.

Proof. By integration by parts

$$\langle [D, F] \rangle = \langle F [D, H] \rangle \tag{6.4}$$

The Schwarz inequality applied to the right side of (6.4) yields

$$\langle [D, F] \rangle \leqslant \langle |F|^2 \rangle^{1/2} \langle \overline{[D, H]}\, [D, H] \rangle^{1/2}$$

To complete the proof, note that when $F = \overline{[D, H]}$, (6.4) becomes

$$\langle [D, \overline{[D, H]}\,] \rangle = \langle \overline{[D, H]}\, [D, H] \rangle \quad \blacksquare$$

We now specialize to the case where H is translation invariant, defined in a periodic box Λ. Let A be a function of one variable and D_j a first-order differential operator in $\phi(j)$. We set

$$\hat{A}(p) = |\Lambda|^{-1/2} \sum_{j \in \Lambda} e^{ip \cdot j} A(\phi(j)) \tag{6.5}$$

and

$$h(p) = \sum_{j \in \Lambda} e^{-ip \cdot j} \langle [D_j, [D_0, H]] \rangle \tag{6.6}$$

Theorem 6.3. In the above situation

$$\langle |\hat{A}(p)|^2 \rangle \geq \langle [D_0, A(\phi(0))] \rangle h(p)^{-1}$$

Proof. The proof follows directly from Lemma 6.2 by setting

$$D = |\Lambda|^{-1/2} \sum_j e^{-ip \cdot j} D_j, \qquad F = \hat{A}(p)$$

because

$$\begin{aligned} \langle [D, F] \rangle &= |\Lambda|^{-1} \sum_j \langle [D_j, A(\phi(j))] \rangle \\ &= \langle [D_0, A(\phi(0))] \rangle \end{aligned}$$

by translation invariance. ■

Note that in subsequent applications we usually pass to the thermodynamic limit, $\Lambda \uparrow \mathbb{Z}^\nu$.

Application 1. In the study of the (Dnhc) ensemble we set

$$D_j = \partial / \partial \phi(j) \tag{6.7}$$

Then

$$h(p) = \Delta(p) [\beta^{-1} + z \langle \cos \delta \phi(0) \rangle]$$

For $A(\phi(j)) = \phi(j)$ or $\sin \phi(j)$, Theorem 6.3 gives the bounds

$$\langle |\hat{\phi}(p)|^2 \rangle \geq 1/h(p)$$

$$\langle |\widehat{\sin \phi}(p)|^2 \rangle \geq \langle \cos \phi(0) \rangle / h(p)$$

respectively, which prove (6.1).

Application 2. The (Dhc) system considered at the beginning of Section 5 is only translation invariant with respect to a sublattice, $4\mathbb{Z}^\nu$.

For $z < 1$ the Hamilton function, H, is

$$(1/2\beta) \sum_{j \in \Lambda} (\nabla \phi)(j)^2 - \sum_{j \in 4\mathbb{Z} \cap \Lambda} \log [1 + z \cos \delta_1 \phi(j)]$$

Let D_j be as in (6.7), $D = |\Lambda|^{-1/2} \sum_j e^{-p \cdot j} D_j$, and

$$F = |\Lambda|^{-1/2} \sum_j e^{ip \cdot j} \phi(j) = \hat{\phi}(p)$$

Then, by Lemma 6.2,

$$\langle |\hat{\phi}(p)|^2 \rangle \geq 1/h(p)$$

where

$$h(p) = \langle [D, [\overline{D, H}]] \rangle$$
$$= \sum_{j,k} e^{ip(k-j)} \left\langle \frac{\partial^2 H}{\partial\phi(k)\partial\phi(j)} \right\rangle$$
$$= \beta^{-1}\Delta(p) + g(p)$$

where

$$g(p) \leqslant (1/4^{\nu})\Delta(p)\left\langle \frac{z\cos[\delta_1\phi(0)]}{1 + z\cos[\delta_1\phi(0)]} + \frac{z^2\sin[\delta_1\phi(0)]^2}{\{1 + z\cos[\delta_1\phi(0)]\}^2} \right\rangle$$

whence $h(p) \leqslant (\beta^{-1} + \gamma)\Delta(p)$, with $\gamma \leqslant 2z/4^{\nu}(1-z)^2$. Thus

$$\langle |\hat{\phi}(p)|^2 \rangle \geqslant (\beta^{-1} + \gamma)^{-1}\Delta(p)^{-1}$$

If we replace the dipoles by renormalized ones, trading z for $\bar{z} = ze^{-\beta/8}$, the same arguments can still be applied. The resulting estimate is

$$\langle |\hat{\phi}(p)|^2 \rangle \geqslant (\beta^{-1} + \bar{\gamma})^{-1}\Delta(p)^{-1} \qquad \text{with } \bar{\gamma} \leqslant \text{const}\,\bar{z}/(1-\bar{z})^2$$

which is (6.1′).

Remark. We recall the upper bound (see Theorem 2.4)

$$\langle |\hat{\phi}(p)|^2 \rangle \leqslant \beta^{-1}\Delta(p)^{-1}$$

[provided $\bar{z} < 1$ in the (Dhc) ensemble]. This estimate and (6.1) or (6.1′) imply that in two or more dimensions

$$\langle |(\partial_1\phi)(p)|^2 \rangle = |e^{ip_1} - 1|^2 \langle |\hat{\phi}(p)|^2 \rangle$$

is discontinuous (though bounded) at $p = 0$. This discontinuity clearly implies that the $(\partial_1\phi)$ two-point correlation function cannot cluster integrably fast. From Section 2, Part 1 we thus conclude that the correlation of two infinitesimal test *dipoles immersed in a background dipole gas cannot decay integrably fast*, i.e., there is *no screening*. (The same is true for the truncated correlation of two test charges, as is easy to verify. The result presumably also holds for the standard, truncated dipole–dipole correlation, but our arguments do not prove this.)

Related, but weaker results have recently been found independently by Park.

Application 3. To recover the classical Mermin theorem, consider a vector-valued field, $\phi(j)$, and a Hamiltonian function

$$H = \sum_{i,j} J(i-j)\phi(i)\phi(j) + V(|\phi(j)|) + \epsilon\phi_1(j) \tag{6.8}$$

We set

$$D_j = \phi_1(j)\,\partial/\partial\phi_2(j) - \phi_2(j)\,\partial/\partial\phi_1(j),$$
$$A = \phi_2(0)$$

Then,

$$h(p) \leqslant \sum_j |1 - \cos p\cdot j|\,|J(j)|\,|\langle\phi(0)\cdot\phi(j)\rangle| + \epsilon$$
$$\leqslant \text{const}\, p^2 + \epsilon$$

provided

$$\sum_j |J(j)|j^2 < \infty, \qquad \langle|\phi(0)|^2\rangle < \infty$$

Theorem 6.3 now implies (taking $\epsilon\downarrow 0$)

$$\langle|\hat\phi(p)|^2\rangle \geqslant \text{const}\langle\phi_1(0)\rangle/p^2$$

which is the assertion of Mermin's theorem. Note that for J's with the property that $\hat J(p)$ is convex, $h(p)$ is bounded below by αp^2, $\alpha > 0$, for small p. For this reason, Mermin's theorem can in general not be used to prove absence of spontaneous magnetization in $\nu \geqslant 3$ dimensions, as was pointed out by J. Bricmont.

Application 4. Consider the Hamilton function H defined in (6.8) with couplings $J(j) = -\beta K(j)$, $\beta > 0$, such that $K(j)$ is reflection positive, i.e.,

$$\sum_{\substack{i,j\in\mathbb{Z}^\nu+(1/2,\ldots,1/2)\\ i_1,j_1>0}} z_i\bar z_j K(i_1+j_1, i_2-j_2, \ldots, i_\nu-j_\nu) \geqslant 0$$

for arbitrary $\{z_i \in \mathbb{C}\}_{i_1>0}$, and $\sum_j K(j) = 0$; see Ref. 14. One example is $K = \Delta$, the finite difference Laplacean.

Let $\phi = (\phi_1, \ldots, \phi_N) \in \mathbb{R}^N$, with *a priori* distribution $d^N\phi$. Let $V \geqslant 0$ be such that

$$\int e^{-V(|\phi|)}e^{\gamma|\phi|^2}d^N\phi < \infty \qquad \text{for all } \gamma > 0$$

The infrared bounds of Refs. 20 and 14 say that

$$\langle|\hat\phi_\alpha(p)|^2\rangle^c \leqslant -\big[\beta\hat K(p)\big]^{-1} \qquad \text{for } p \neq 0 \tag{6.9}$$

for all $\alpha = 1, \ldots, N$. Note that for $\epsilon > 0$ (in a finite, periodic box, Λ)

$$\langle|\hat\phi_\alpha(p)|^2\rangle^c = \langle|\hat\phi_\alpha(p)|^2\rangle \qquad \text{for } \alpha \geqslant 2 \tag{6.10}$$

whereas

$$\langle|\hat\phi_1(p)|^2\rangle^c = \langle|\hat\phi_1(p)|^2\rangle + M(\epsilon, \beta)^2|\Lambda|\delta_{0,p} \tag{6.11}$$

with $M(\epsilon, \beta) = \langle\phi_1(0)\rangle$.

Let $I(\nu, K) = \int_{|p_i| \leqslant \pi} d^\nu p \hat{K}(p)^{-1}$. For $K = \Delta$, $-\hat{K}(p) = \Delta(p) \approx p^2$, so that $I(2, \Delta) = \infty$, but $I(\nu, \Delta) < \infty$, for $\nu \geqslant 3$.

By taking the thermodynamic limit (supposed to be ergodic) and integrating (6.10) and (6.11) we obtain from (6.9)

$$\langle \phi_\alpha(0)^2 \rangle \leqslant \beta^{-1} I(\nu, K) \qquad \text{for } \alpha \geqslant 2 \tag{6.12}$$

and

$$\langle \phi_1(0)^2 \rangle \leqslant \beta^{-1} I(\nu, K) + M(\epsilon, \beta)^2 \tag{6.13}$$

Next, we apply Theorem 6.3 with $D_j = \partial / \partial \phi_2(j)$,

$$F = \hat{A}(p) = |\Lambda|^{-1/2} \sum_{j \in \Lambda} e^{ip \cdot j} \phi_2(j)$$

By (6.6) and (6.8) and our choice of $J(j)$,

$$h(p) = -\beta K(p) + \left\langle \frac{\partial^2 V(|\phi(0)|)}{\partial \phi_2(0)^2} \right\rangle$$

Theorem 6.3 then gives

$$\langle |\phi_2(p)|^2 \rangle \geqslant \left\langle \frac{\partial \phi_2(0)}{\partial \phi_2(0)} \right\rangle h(p)^{-1} = h(p)^{-1}$$

$$= \left\{ -\beta \hat{K}(p) + \left\langle \frac{\partial^2 V[|\phi(0)|]}{\partial \phi_2(0)^2} \right\rangle \right\}^{-1} \tag{6.14}$$

Comparing this with (6.9) and (6.10) we conclude that

$$\left\langle \frac{\partial^2 V[|\phi(0)|]}{\partial \phi_2(0)^2} \right\rangle \geqslant 0 \tag{6.15}$$

This inequality is stable undertaking the thermodynamic limit. Now let

$$V(|\phi|) = \lambda |\phi|^4 - \frac{\sigma}{2} |\phi|^2 + \text{const} \tag{6.16}$$

Then

$$\frac{\partial^2 V}{\partial \phi_2^2} = 4\lambda \phi^2 + 8\lambda \phi_2^2 - \sigma$$

Thus

$$\left\langle \frac{\partial^2 V[|\phi(0)|]}{\partial \phi_2(0)^2} \right\rangle = 4\lambda \left[\langle \phi_1(0)^2 \rangle + \sum_{\alpha=2}^{N} \langle \phi_\alpha(0)^2 \rangle \right]$$

$$+ 8\lambda \langle \phi_2(0)^2 \rangle - \sigma$$

By (6.12) and (6.13) the right-hand side is bounded above by $4\lambda M(\epsilon, \beta)^2 + 4(N+2)\lambda\beta^{-1}I(\nu,K) - \sigma$. This and (6.15) yield

$$M(\epsilon, \beta) \geq \left[\frac{\sigma - 4(N+2)\lambda\beta^{-1}I(\nu,K)}{4\lambda} \right]^{1/2}$$
$$= \frac{1}{2}\left(\frac{\sigma}{\lambda}\right)^{1/2} - \text{const } \beta^{-1} \tag{6.17}$$

if $I(\nu,K)$ is *finite*. As $\beta \to \infty$

$$M(\epsilon, \infty) \geq \frac{1}{2}\left(\frac{\sigma}{\lambda}\right)^{1/2} \tag{6.18}$$

The right-hand side of (6.18) is precisely the value of the spontaneous magnetization predicted by the naive Goldstone picture!

These arguments can be extended in several ways.

(A) Let $V(|\phi|)$ be an arbitrary, positive polynomial. Applying Gaussian domination to bound $\langle\phi_\alpha(0)^{2m}\rangle^c$ by $O(\beta^{-m})$, provided $I(\nu,K) < \infty$, see Refs. 20 and 14, we conclude that

$$\left\langle \frac{\partial^2 V(|\phi(0)|)}{\partial\phi_2(0)^2} \right\rangle = \frac{\partial^2 V}{\partial\phi_2^2}\left[|(M(\epsilon, \beta), 0, \ldots, 0)|\right] - \text{const } \beta^{-1}$$

This and (6.15) generally give as a lower bound for $M(\epsilon, \beta)$ the smallest value predicted by the naive Goldstone picture, up to $O(\beta^{-1})$ corrections.

(B) Let $\nu = 2$, pass to the continuum limit and choose

$$-K(p) = (p^2)^\lambda, \qquad \tfrac{3}{4} < \lambda < 1$$

Then

$$-\int_{|p|\leq 1} d^2p K(p)^{-1} < \infty$$

Let $V(|\phi|) = \lambda : (\phi\cdot\phi)^2 : - \frac{1}{2}\sigma : \phi\cdot\phi :$, where $:-:$ is Wick order with respect to $-(\beta K)^{-1}$. This defines a Euclidean field theory model which has been constructed and shown to exhibit spontaneous magnetization for sufficiently large β in Ref. 42. The arguments described above can be applied to this model, with some obvious changes. The analog of (6.17) is

$$M(\epsilon, \beta) \geq \frac{1}{2}\left[\frac{\sigma + \delta(\beta)}{\lambda}\right]^{1/2} \tag{6.19}$$

for some $\delta(\beta) \geq 0$(!). Here we have used that $\partial/\partial\phi_\alpha(0) : \phi_\alpha(0)^n : = n : \phi_\alpha(0)^{n-1} :$, and $\langle : \phi_\alpha(0)^2 : \rangle^c \leq 0$, for our choice of Wick order (see, e.g., Ref. 20).

Thus, for $\sigma > 0$ there is a nonzero spontaneous magnetization, in accordance with the Goldstone picture.

7. PHASE TRANSITIONS AND SPONTANEOUS ORIENTATION OF DIPOLES IN HARD CORE DIPOLE LATTICE GASES

In this section we show that hard core dipole lattice gases in three or more dimensions undergo phase transitions (at small temperatures, as the dipole activity is varied), and we exhibit equilibrium states with spontaneous orientation of dipoles and broken translational invariance, at small temperature and large activity.

Our proofs of these results are based on reflection positivity (RP), established in Appendix A. We use RP as a means for establishing infrared bounds from which our results follow in a fairly standard fashion.[20,6]

In the case of hard core dipole gases in two or more dimensions with the property that each dipole only has finitely many possible orientations one can combine RP with a Peierls argument[5,14] to establish the results mentioned above. (We shall not give full details, which the reader can easily reconstruct from Refs. 5, 6, 14 and some hints that we shall sketch.)

This section is organized as follows: in Section 7.1 we specify the dipole potentials and ensembles considered in the remainder and briefly review RP.

In Section 7.2 we establish some important properties of dipole potentials and exhibit the ground states of dipole gases. In a sense Section 7.2 is the technical core of Section 7. It should be read after a first glance at the later sections. (We thank B. Simon for some hints concerning the material in Section 7.2.)

In Section 7.3 we establish the required infrared bounds which, in Section 7.4, are applied to prove existence of a phase transition. (We note that for short range dipole potentials, one can use a high-temperature expansion to prove uniqueness for small β. In the case of long-range potentials, inequalities of Section 2 give absence of ordering in the two-point function for activity $z < 1$ and uniqueness for $z < \beta^{-1}$.)

A few hints concerning the Peierls argument for two-dimensional, discrete dipole gases are given in Section 7.5.

7.1. Dipole Ensembles and Reflection Positivity

Reflection positivity[13,14] for Coulomb and dipole gases is established in Appendix A. Here we only recall the basic facts. We consider the following class of dipole gases: each site of $\mathbb{Z}^\nu$ may either be empty or occupied by one dipole with some dipole moment $q \in \mathbb{R}^\nu$. The *a priori* law of the dipole moment is given by a probability measure $d\rho(q)$ on $\mathbb{R}^\nu$. The potential energy of a dipole at site i and one at site j is given by

$$(q_i, W(i-j)q_j) \tag{7.1}$$

where $W(j)$ is some $\nu \times \nu$ matrix whose general properties are discussed later, and q_i, q_j are in the support of $d\rho$. We have the following examples of dipole potentials, W, in mind.

(a) Let $e_1, \ldots, e_\nu$ be unit lattice vectors in the direction of the coordinate axes. We define $W(j), j \in \mathbb{Z}^\nu$, by the equation

$$(q, W(j)q') = -(\delta_q \delta_{q'} C)(j) \tag{7.2}$$

where $(\delta_a F) \equiv F(x + a) - F(x)$, and C is some potential on

$$\mathbb{L}_\rho = \{ j + q : j \in \mathbb{Z}^\nu, q \in \operatorname{supp} d\rho \cup \{0\} \}$$

Later it will be necessary to constrain $\operatorname{supp} d\rho$ to a hypercube centered at 0 with sides parallel to the coordinate axes of length $\leqslant 1$.

(b) Let $\hat{q} = \lambda q / |q|, 0 < \lambda < 1/2$, and define $W(j)$ by

$$(q, W(j)q') = -|q|\,|q'|(\delta_{\hat{q}} \delta_{\hat{q}'} C)(j) \tag{7.3}$$

with C some potential on

$$\mathbb{L}_\lambda = \{ j + a : j \in \mathbb{Z}^\nu, |a| = \lambda \}$$

(c)

$$W(j) = \left[W_{\alpha\gamma}(j) \right]_{\alpha,\gamma=1}^{\nu} \tag{7.4}$$

with $W_{\alpha\gamma}(j) = (\partial^2/\partial x^\alpha \partial x^\gamma C)(j)$, where C is some potential on $\mathbb{R}^\nu$. A typical example of a potential C is

$$C(x) = \begin{cases} (-\Delta + \epsilon)^{-1}(x) & \text{for } |x| > 1 - 2\lambda \\ \text{const} & \text{for } |x| \leqslant 1 - 2\lambda \end{cases} \tag{7.5}$$

$\epsilon \geqslant 0$, $0 < \lambda < 1/2$.

Note that by adding a suitable bounded function g supported in $\{x : |x| \leqslant 1 - 2\lambda\}$ one can achieve that $(C + g)(x)$ is of positive type and regular near $x = 0$. Thus, the tools of Sections 2–6 are available (they are, however, quite inessential in this section).

Henceforth we shall restrict our considerations to the dipole potentials introduced in (c) which are restrictions of regularized continuum dipole potentials to $\mathbb{Z}^\nu$. Our methods apply equally well to the sort of potential defined in (a) and (b) which we studied throughout most of Sections 5 and 6.

Let Λ be a periodic box in $\mathbb{Z}^\nu$, viewed as the restriction of a periodic box in $\mathbb{R}^\nu$ to $\mathbb{Z}^\nu$, and let W_Λ be an infrared regularized and periodized version of W on Λ, with $W_\Lambda \to W$, as $\Lambda \uparrow \mathbb{Z}^\nu$, e.g., in the quadratic form sense. If w is given in terms of a scalar potential C, as in (a)–(c), and C is given by (7.5) then $C_\Lambda = (-\Delta^\Lambda + \epsilon^\Lambda)^{-1}(x)$, where Δ^Λ is the Laplacean with periodic boundary conditions at $\partial\Lambda$, and $\{\epsilon^\Lambda > 0\}$ is a sequence of infrared regulators. The details of how one chooses the periodic approximation, W_Λ, to W are quite unimportant, but we shall require that the W_Λ's be reflection positive; see inequality (7.13) below.

The Hamilton function in the periodic box, Λ, is given by

$$H_\Lambda = (\beta/2) \sum_{i,j\in\Lambda} (q_i, W_\Lambda(i-j)q_j)$$
$$\equiv (\beta/2)(q, W_\Lambda q) \tag{7.6}$$

The Gibbs expectations, $\langle\cdot\rangle \equiv \langle\cdot\rangle_\Lambda$, is given by the probability measure

$$Z_\Lambda^{-1} e^{-H} \Lambda d\rho(q) \tag{7.7}$$

with $d\rho(q) = \prod_j d\rho(q_j)$, and Z_Λ the obvious normalization factor.

Next, we recall sufficient conditions on W or C which guarantee that the Gibbs state, $\langle\cdot\rangle$, is *reflection positive*.[14]

For $q \in \mathbb{R}^\nu$, define

$$R_\alpha q = R_\alpha(q^1, \ldots, q^\alpha, \ldots, q^\nu) = (-q^1, \ldots, q^\alpha, \ldots, -q^\nu) \tag{7.8}$$

Let π_α be a pair of hyperplanes perpendicular to the α direction, midway in between two lattice planes and bisecting Λ into two pieces Λ_+ and Λ_- of equal size. Let r_α denote reflection of sites in Λ at π_α. Clearly $r_\alpha\Lambda_- = \Lambda_+$. We define

$$(\theta_\alpha q)_j = R_\alpha q_{r_\alpha j} \tag{7.9}$$

Let $q_\pm = \{q_j\}_{j\in\Lambda_\pm}$. If A is a function of q_+ we set

$$(\theta_\alpha A)(q_-) = A(\{\theta_\alpha q_j\}_{j\in\Lambda_-}) \tag{7.10}$$

In the context of dipole gases the most natural definition of reflection positivity (RP) of a Gibbs state, $\langle\cdot\rangle_\Lambda$, is as follows.

Definition 7.1. The expectation $\langle\cdot\rangle_\Lambda$ is said to satisfy RP iff, for an arbitrary function A of q_+,

$$\langle \overline{\theta_\alpha A(q_-)} A(q_+) \rangle_\Lambda \geq 0 \tag{7.11}$$

for all $\alpha = 1, \ldots, \nu$.

We now give a sufficient condition on W_Λ and $d\rho$ for (7.11) to hold.

Proposition 7.2. Let $d\rho$ be chosen such that

$$d\rho(R_\alpha q) = d\rho(q) \qquad \text{for all } \alpha \tag{7.12}$$

Let $Q_+ = \{Q_j\}_{j\in\Lambda_+}$ be an arbitrary $\mathbb{R}^\nu$-valued function on Λ_+ (i.e., $Q_j = 0$, for all $j \in \Lambda_-$). Assume that

$$-\sum_{i,j} (Q_i, W(i - r_\alpha j) R_\alpha Q_j) \geq 0 \tag{7.13}$$

for all such Q_+, all $\alpha = 1, \ldots, \nu$. Then $\langle\cdot\rangle_\Lambda$ satisfies RP.

We do not give the proof of Proposition 7.2, which is an adaptation of arguments in Ref. 14, Section 3.D (see also Section 5 of Ref. 6). Instead, we now suppose that W is defined as in (7.2), (7.3), or (7.4).

Let f be an arbitrary scalar function on

$$\Lambda_+ \cap L_\rho \qquad \text{if } W \text{ is as in (7.2)}$$

or on

$$\Lambda_+^w \cap L_\lambda \qquad \text{if } W \text{ is as in (7.3), (7.4)} \tag{7.14}$$

(Note that Λ_+ is, here, considered a subset of $\mathbb{R}^\nu$.) Let $\theta_\alpha' f(x) \equiv f(r_\alpha x)$.

In the following we usually suppress the subscript Λ, unless a specific context requires adding it.

Proposition 7.3 [Reflection Positivity (RP)]. Suppose that W is defined as in (7.2), (7.3), or (7.4). Let C be of positive type, and

$$C(r_\alpha x, r_\alpha y) = C(x, y)$$
$$(\theta_\alpha' f, Cf) \geqslant 0 \tag{7.15}$$

for all f's as in (7.14), for all $\alpha = 1, \ldots, \nu$. Then condition (7.13) of Proposition 7.2 holds, and $\langle \cdot \rangle$ is RP in the sense of Definition 7.1.

The *proof* of Proposition 7.3 is given in Appendix A. [A direct verification of the fact that (7.15) implies (7.13) can also be found by modifying arguments in Section 5 of Ref. 6.] We emphasize that the hypotheses of Proposition 7.3 are satisfied for C as in (7.5). In the remainder of Section 7 we limit our attention to dipole potentials of the form specified in (c), (7.4), for some C satisfying (7.15).

7.2. Properties of the Dipole Potential, Ground States of Dipole Gases

Consider the $\nu \times \nu$ matrix $W_{\alpha\gamma}^c(x)$ given by

$$W_{\alpha\gamma}^c(x) = -(\partial^2/\partial x^\alpha \partial x^\gamma C)(x) \qquad \text{for all } x \in \mathbb{R}^\nu \tag{7.16}$$

where C is a translation-invariant quadratic form, and $C(x, y) \equiv C(x - y)$ is its integral kernel. We assume that the hypotheses of Proposition 7.3 (RP) are satisfied.

For this it suffices, e.g., that, for $|x| > 1 - 2\lambda$, $0 < \lambda < 1/2$, $C(x)$ has a Källen–Lehmann spectral representation,

$$C(x) = \int d\mu(a)(-\Delta + a)^{-1}(x) \qquad \text{for } |x| > 1 - 2\lambda \tag{7.17}$$

where $d\mu$ is some measure on $[0, \infty)$ with $\int (a+1)^{-1} d\mu(a) < \infty$, and Δ is the Laplacean with periodic boundary conditions at $\partial\Lambda$. Then, for a suitable reflection-invariant continuation of $C(x)$ to $\{x : |x| \leqslant 1 - 2\lambda\}$ conditions

(7.15) hold. Moreover, $C(x)$ can be chosen to be of positive type. (All this is easy to check.)

The Fourier transform of $W^c_{\alpha\gamma}(x)$ [see (7.16)] is

$$\hat{W}^c_{\alpha\beta}(k) = k_\alpha k_\gamma \hat{C}(k) \tag{7.18}$$

with $\hat{C}(k)$ the Fourier transform of C. Let $P(k)$ denote the orthogonal projection on $k/|k|$, i.e.,

$$P_{\alpha\gamma}(k) = |k|^{-2} k_\alpha k_\gamma \tag{7.19}$$

By (7.18)

$$\hat{W}^c(k) = \hat{D}(k)P(k) \quad \text{with } \hat{D}(k) = k^2\hat{C}(k) \tag{7.20}$$

Using the spectral representation (7.17), it is not hard to show that one can choose C such that

(A) C satisfies (7.15), so that $\langle \cdot \rangle$ satisfies RP, in the sense of Definition 7.1;
(B) $\hat{D}(k) > 0$, for all k;
(C) $\hat{D}(k)$ falls off rapidly, as $|k| \to \infty$.

Let $W(j)$ be the restriction of W^c to the lattice, i.e., $j \in \mathbb{Z}^\nu$. The Poisson summation formula expresses the Fourier transform of W in terms of $\hat{W}^c$:

$$\begin{aligned}\hat{W}(k) &= \sum_{m \in \mathbb{Z}^\nu} \hat{W}^c(k + 2\pi m) \\ &= \sum_{m \in \mathbb{Z}^\nu} \hat{D}(k + 2\pi m)P(k + 2\pi m)\end{aligned} \tag{7.21}$$

Let l be an arbitrary vector on the unit sphere, $S^{\nu-1}$, of $\mathbb{R}^\nu$. Then there exists a sequence $\{m_r\}_{r=1}^\infty \subset \mathbb{Z}^\nu$ such that

$$\frac{k + 2\pi m_r}{|k + 2\pi m_r|} \to l \qquad as \quad r \to \infty$$

To see this we propose to choose $\{m_r\}$ such that $|m_r| \to \infty$, as $r \to \infty$. In this case

$$|k|/|k + 2\pi m_r| \to 0 \quad \text{and} \quad \frac{|2\pi m_r|}{|k + 2\pi m_r|} \to 1$$

for all $k \in B$, the first Brillouin zone (i.e., the dual of $\mathbb{Z}^\nu$). Thus, it is enough to show that $m_r/|m_r| \to l$, as $r \to \infty$. This follows by an obvious density argument. Finally, since $\lambda m_r/|\lambda m_r| = m_r/|m_r|$, $\lambda = 1, 2, 3, \ldots$, $\{m_r\}$ can clearly be chosen such that $|m_r| \to \infty$, as $r \to \infty$. This proves our contention. Next, we note that if $\{l_s\}_{s=1}^\infty$ is a dense set of points on $S^{\nu-1}$, and $\{c_s\}_{s=1}^\infty$ is a summable sequence of positive numbers then $\sum c_s P(l_s)$ is a strictly positive $\nu \times \nu$ matrix. By condition (B), $\hat{D}(k)$ is positive, for all k.

Thus

$$\hat{W}(k) = \sum_{m \in \mathbb{Z}^\nu} \hat{D}(k + 2\pi m) P\left(\frac{k + 2\pi m}{|k + 2\pi m|} \right)$$

is *strictly positive*, for all $k \in B$. [In contrast, $\hat{W}^c(k) = \hat{D}(k)P(k)$ is *singular* for all $k \in \mathbb{R}^\nu$, $\nu \geqslant 2$.]

Next, we claim that $\hat{W}(k)$ is independent of the way $C(x)$ is regularized on $\{x : |x| < 1\}$, up to a constant multiple of the identity. For, $W_{\alpha\gamma}(j) = W^c_{\alpha\gamma}(j)$, for $j \in \mathbb{Z}^\nu$ with $|j| \neq 0$. Moreover, since $C(x)$ has been assumed to be symmetric under interchanging x^α and x^γ, $W_{\alpha\gamma}(0) = c\delta_{\alpha\beta}$, for some constant $c \geqslant 0$. The Fourier transform of $W^c_{\alpha\gamma}(j)(1 - \delta_{j,0})$ is clearly regularization independent, whereas the Fourier transform of $W_{\alpha\gamma}(0)\delta_{j,0}$ is equal to $c\delta_{\alpha\gamma}$.

We are now going to choose c in a way that is convenient for our purposes. Let

$$k = (0, k_2, \ldots, k_\nu) \tag{7.22}$$

Then

$$\hat{W}(k) = \begin{bmatrix} \Delta_k & 0 \cdots 0 \\ 0 & \\ \vdots & \hat{W}^{(1)}(k) \\ 0 & \end{bmatrix} \tag{7.23}$$

with

$$\Delta_k = \sum_{m \in \mathbb{Z}^\nu} (2\pi m_1)^2 \hat{C}(k + 2\pi m) \tag{7.24}$$

To prove this we recall that

$$\hat{W}_{\alpha\gamma}(k) = \sum_{m \in \mathbb{Z}^\nu} \nu(k_\alpha + 2\pi m_\alpha)(k_\gamma + 2\pi m_\gamma)\hat{C}(k + 2\pi m)$$

For $\alpha = 1$ or $\gamma = 1$ we have

$$\hat{W}_{\alpha\gamma}(k) = \sum_{m \in \mathbb{Z}^\nu} 2\pi m_1(k_\mu + 2\pi m_\mu)\hat{C}(k + 2\pi m) \tag{7.25}$$

Since $\hat{C}$ is even, $\hat{C}(k + 2\pi m)$ is even in m_1 for k as in (7.22) so that by (7.25)

$$W_{\alpha\gamma}(k) = 0 \qquad \text{for} \quad \alpha = 1 \quad \textit{or} \quad \gamma = 1 \quad \text{and} \quad \alpha \neq \gamma$$

and $W_{11}(k) \equiv \Delta_k$ is given by (7.24). Analogous statements hold when 1 is replaced by any $\alpha = 2, \ldots, \nu$.

Next, let $\pi^{(\alpha)} \in B$ be the vector with components

$$\pi^{(\alpha)}_\gamma = \pm\pi(1 - \delta_{\alpha\gamma})$$

and set

$$\Delta = \Delta_{\pi(\alpha)} \tag{7.26}$$

(which is independent of α). Let

$$\hat{W}^0(k) \equiv \hat{W}(k) - \Delta 1 \tag{7.27}$$

$\hat{W}^0$ is clearly regularization independent.

Lemma 7.4. For all $k \in B$, $\hat{W}^0(k) \geqslant 0$, as a $\nu \times \nu$ matrix.

Proof. Let

$$R(j) \equiv \prod_{\alpha=1}^{\nu} R_\alpha^{j^\alpha} \tag{7.28}$$

where R_α is the involution ("dipole reflection") defined in (7.8).

Let h be an $\mathbb{R}^\nu$-valued function on the periodic box Λ. Let $h_\pm = h\chi_{\Lambda_\pm}$, where $\Lambda_\pm$ are the two halves of Λ separated by the pair of hyperplanes π_α, and $\chi_{\Lambda_\pm}$ the characteristic functions of $\Lambda_\pm$. Let θ_α be defined as in (7.9). By inequality (7.13) we have the following Schwarz inequality:

$$\begin{aligned}
-(h, W^0 h) &= -(h_+, W^0 h_+) - (h_-, W^0 h_-) - (h_+, W^0 h_-) - (h_-, W^0 h_+)\\
&\leqslant -(h_+, W^0 h_+) - (h_-, W^0 h_-)\\
&\quad + 2\left[-(h_+, W^0\theta_\alpha h_+)\right]^{1/2}\left[-(h_-, W^0\theta_\alpha h_-)\right]^{1/2}\\
&\leqslant -(h_+, W^0 h_+) - (h_+, W^0\theta_\alpha h_+)\\
&\quad - (h_-, W^0 h_-) - (h_-, W^0\theta_\alpha h_-)\\
&= -\left[\tfrac{1}{2}(h_+ + \theta_\alpha h_+, W(h_+ + \theta_\alpha h_+))\right.\\
&\quad \left. + (h_- + \theta_\alpha h_-, W^0(h_- + \theta_\alpha h_-))\right]
\end{aligned} \tag{7.29}$$

If the sides of Λ have length 2^{s_α}, $s_\alpha = 1, 2, 3, \ldots$, $\alpha = 1, \ldots, \nu$, we may iterate inequality (7.29) for a new choice of π_α in both terms on the right-hand side of (7.29). Proceeding in this way, with all possible choices for π_α and α, we arrive at the inequality

$$-(h, W^0 h) \leqslant \frac{1}{|\Lambda|} \sum_{i \in \Lambda} -(h^{(i)}, W^0 h^{(i)}) \tag{7.30}$$

where

$$h^{(i)}(j) = R(j - i)h(i) \tag{7.31}$$

[If the lengths of the sides of Λ are even, but not powers of 2, (7.30) follows from (7.29) and an optimization argument of the sort used in Ref. 14, Section 3.]

We now show that each term on the right-hand side of (7.30) vanishes. This follows from

$$\sum_{i\in\Lambda} W^0(l-i)R(i)=0 \tag{7.32}$$

This identity is proven by a computation:

$$\begin{aligned}\left[\sum_{i\in\Lambda} W^0(l-i)R(i)\right]_{\alpha\gamma} &= \sum_{i\in\Lambda} W^0(l-i)_{\alpha\gamma}R(i)_{\gamma\gamma}\\ &= \sum_{i\in\Lambda} W^0(l-i)_{\alpha\gamma}(-1)^{\sum_{\mu\neq\gamma} i^\mu}\\ &= (-1)^{\sum_{\mu\neq\gamma} l^\mu}\,\hat{W}^0(\pi^{(\gamma)})\alpha\gamma=0\end{aligned}$$

with $\pi^{(\gamma)}$ as in (7.26), by (7.28) and (7.27). Since this holds for all α and γ, (7.32) is proven. ■

Remark. Identity (7.32) shows that any dipole configuration $\mathring{q}$ defined by

$$\mathring{q}_j = R(j)q \quad \textit{for some } q\in\mathbb{R} \tag{7.33}$$

has vanishing energy density (with respect to the dipole potential W^0). By Lemma 7.4, $\mathring{q}$ is therefore a *ground-state configuration.*

We now show that, in certain cases, the inequality of Lemma 7.4 can be improved. For this purpose one must first pass to the limit $\Lambda=\mathbb{Z}^\nu$. Let l be some unit vector in $\mathbb{R}^\nu$. Let $W^0(m,\mathbf{k})$, $\mathbf{k}=(k_2,\ldots,k_\nu)$, be the partial Fourier transform of $W^0(j)$ with respect to $(j^2,\ldots,j^\nu)$; $j^1\equiv m\in\mathbb{Z}$.

We set

$$V_{\mathbf{k}}(m) = -\big(l,\tilde{W}^0(m,\mathbf{k})R_1 l\big) \tag{7.34}$$

Then inequality (7.13) or (7.15) clearly implies the following inequality for V:

$$\sum_{m,n\geqslant 1} \bar{z}_M V_{\mathbf{k}}(m+n-1)z_n \geqslant 0 \tag{7.35}$$

for arbitrary $\{z_m\}_{m=1}^\infty\subset\mathbb{C}$, i.e., $V_{\mathbf{k}}$ is a *translation-invariant: reflection-positive* two-point function. Such two-point functions are known to have the following spectral representation:

$$V_{\mathbf{k}}(m)=\int_{-1}^{1} d\rho(\lambda,\mathbf{k})\lambda^{|m|-1} \qquad \text{for } |m|>1 \tag{7.36}$$

See, e.g., Section 5 of Ref. 6.

The Fourier transform of $V_{\mathbf{k}}(m)$ in m is therefore given by

$$\hat{V}_{\mathbf{k}}(k_1) = c + 2\int_{-1}^{1} d\rho(\lambda,\mathbf{k})(\cos k_1 - \lambda)\left(1 + \lambda^2 - 2\lambda\cos k_1\right)^{-1} \quad (7.37)$$

for some constant $c = c_{\mathbf{k}}$. (See Section 5 of Ref. 6, discussion of Model 5.3.) From (7.37) follow

$$\begin{aligned} &\hat{V}_{\mathbf{k}}(0) > \hat{V}_{\mathbf{k}}(k_1) \qquad \text{for } k_1 \neq 0 \\ &\hat{V}_{\mathbf{k}}(k_1) > \hat{V}_{k}(\pi) \qquad \text{for } k_1 \neq \pm\pi \end{aligned} \quad (7.38)$$

provided

$$V_{\mathbf{k}}(m) \neq 0 \qquad \text{for some } |m| \geqslant 1$$

and

$$V_{\mathbf{k}}(m) \to 0 \qquad \text{as } |m| \to \infty \quad (7.39)$$

For $V_{\mathbf{k}}(m)$ as defined in (7.34), (7.39) is checked easily. Thus, we have the following proposition.

Proposition 7.5. (1) $\hat{W}^0_{\alpha\alpha}(k) > \hat{W}^0_{\alpha\alpha}(\pi^{(\alpha)}) = 0$ [$\pi^{(\alpha)}$ as in (7.26)], if $k_\mu \neq \pm\pi(1-\delta_{\alpha\mu})$ for some μ. (2) $\hat{W}^0_{\alpha\alpha}(k) < \hat{W}^0_{\alpha\alpha}((0,\ldots,0,\underset{\alpha}{\pm\pi},0,\ldots,0)$, if $k_\mu \neq \pm\pi\delta_{\alpha\mu}$, for some μ.

Proof. By (7.38), (7.34) and the definition (7.8) of R_1, we have the following:

$$\begin{array}{lll} \text{(i)} & -\hat{W}^0_{11}(0,\mathbf{k}) > -\hat{W}^0_{11}(k_1,\mathbf{k}) & \text{for } k_1 \neq 0 \\ \text{(ii)} & -\hat{W}^0_{11}(k_1,\mathbf{k}) > -\hat{W}^0_{11}(\pi,\mathbf{k}) & \text{for } k_1 \neq \pm\pi \\ \text{(iii)} & \hat{W}^0_{\alpha\alpha}(0,\mathbf{k}) > \hat{W}^0_{\alpha\alpha}(k_1,\mathbf{k}) & \text{for } k_1 \neq 0 \\ \text{(iv)} & \hat{W}^0_{\alpha\alpha}(k_1,\mathbf{k}) > \hat{W}^0_{\alpha\alpha}(\pi,\mathbf{k}) & \text{for } k_1 \neq \pm\pi \end{array}$$

Next, using first (i) and then (iv) with 1 and α interchanged, we see that

$$\begin{aligned} &\hat{W}^0_{11}(k_1,k_2,\ldots,k_\alpha,\ldots,k_\nu) \\ &\quad \geqslant \hat{W}^0_{11}(0,k_2,\ldots,k_\alpha,\ldots,k_\nu) \\ &\quad \geqslant \hat{W}^0_{11}(0,k_2,\ldots,\pi,\ldots,k_\nu) \geqslant \cdots \\ &\quad \geqslant \hat{W}^0_{11}(\pi^{(1)}) = 0 \end{aligned}$$

and if $k_\mu \neq \pm\pi[1-\delta_{1\mu}]$, for some μ, at least one of the inequalities is strict, so that

$$\hat{W}^0_{11}(k) > 0$$

and exchanging 1 and α,

$$\hat{W}^0_{\alpha\alpha}(k) > \hat{W}_{\alpha\alpha}(\pi^{(\alpha)}) = 0$$

if $k_\mu \neq \pm\pi[1-\delta_{\alpha\mu}]$, for some μ.

This proves (1). To prove (2) we apply (ii) and subsequently (iii) with 1 and α interchanged. This gives

$$\begin{aligned}\hat{W}^0_{11}(k) &\leqslant \hat{W}^0_{11}(\pi, k_2, \ldots, k_\alpha, \ldots, k_\nu)\\ &\leqslant W^0_{11}(\pi, k_2, \ldots, 0, \ldots, k_\nu)\\ &\leqslant \cdots \leqslant \hat{W}^0_{11}(\pi, 0, \ldots, 0)\end{aligned}$$

and if $k_\mu \neq \pm\pi\delta_{1\mu}$, for some μ, at least one inequality is strict. ■

We now consider the specific energy of some periodic, two-dimensional dipole configurations, arising in estimating contour probabilities in a Peierls argument; see Section 7.5. (The subsequent inequalities extend to arbitrary dimensions, $\nu > 2$. In order to economize on notations we only consider $\nu = 2$.)

(I) We define a dipole configuration, q^{I}, by $q_0^{(\mathrm{I})} = (1,0), q_{(0,1)}^{(\mathrm{I})} = (-1, 0), (\uparrow\downarrow)$; for general $j \in \mathbb{Z}^2$, let $q_j^{(\mathrm{I})}$ be given by consecutive reflections of $(q_0^{(\mathrm{I})}, q_{(0,1)}^{(\mathrm{I})})$ in lines parallel to the 1 and 2 axes (between sites), i.e.,

$$q^{(\mathrm{I})}_{(j_1,j_2)} = (1,0) \qquad \text{if } j_2 \textit{ is even}$$

$$q^{(\mathrm{I})}_{(j_1,j_2)} = (-1,0) \qquad \text{if } j_2 \text{ is odd}$$

Given an arbitrary pair $(q_0, q_{(0,1)})$ of vectors in $\mathbb{R}^2$, we set

$$q_{(j_1,j_2)} = R_1^{j_1} R_2^{j_2} q_0 \qquad \text{for } j_2 = 4n-1, 4n$$

and

$$q_{(j_1,j_2)} = R_1^{j_1} R_2^{j_2-1} q_{(0,1)} \qquad \text{for } j_2 = 4n+1, 4n+2$$

$n \in \mathbb{Z}$. The $\mathbb{R}^2$-valued function, q, on $\mathbb{Z}^2$ obtained in this way is called "periodic extension of $(q_0, q_{(0,1)})$."

(II) $q^{(\mathrm{II})}$ is defined as the periodic extension of

$$q_0^{(\mathrm{II})} = (1,0), \qquad q_{(0,1)}^{(\mathrm{II})} = (1,0) \qquad (\uparrow\uparrow)$$

(III) $q^{(\mathrm{III})}$ is defined as the periodic extension of

$$q_0^{(\mathrm{III})} = (1,0), \qquad q_{(0,1)}^{(\mathrm{III})} = (0, \pm 1) \qquad (\uparrow\leftarrow \text{ or } \uparrow\rightarrow)$$

(IV) $q^{(\mathrm{IV})}$ is obtained from

$$q_0^{(\mathrm{IV})} = (1,0), \qquad q_{(0,1)}^{(\mathrm{IV})} = (0,0) \qquad (\uparrow\varnothing)$$

(V) $q^{(\mathrm{V})}$ is obtained from

$$q_0^{(\mathrm{V})} = (0,1), \qquad q_{(0,1)}^{(\mathrm{V})} = (0,0) \qquad (\rightarrow\varnothing)$$

(VI) Finally $q^{(\mathrm{VI})} \equiv 0$ is the periodic extension of

$$q_0^{(\mathrm{VI})} = (0,0) = q_{(0,1)}^{(\mathrm{VI})} \qquad (\varnothing\varnothing)$$

We now introduce the specific energies ϵ^r, $r = \mathrm{I}, \ldots, \mathrm{VI}$, of these configurations:

$$\epsilon^r = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \sum_{i,j \in \Lambda} \left(q_i^{(r)}, W_\Lambda^0(i-j) q_j^{(r)} \right) \tag{7.40}$$

Proposition 7.5 has the following corollary.

Corollary 7.6. (1) $\epsilon^{\mathrm{I}} = \epsilon^{\mathrm{VI}} = 0$; (2) $\epsilon^r > 0$, for $r = \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$.

Proof. By Fourier transformation of the right-hand side of (7.40), $\epsilon^{\mathrm{I}} = \hat{W}^0((0, \pm\pi)) = 0$, $\epsilon^{\mathrm{VI}} = 0$, because $q^{\mathrm{VI}} \equiv 0$.

For $r = \mathrm{II}, \mathrm{IV}, \mathrm{V}$, the proof of (2) is a simple variant of the arguments used in the discussion of Model 5.3, Section 5 of Ref. (6): by Fourier transformation and the fact that $q^{(r)}$ is periodic, ϵ^r is easily seen to be of the form

$$\epsilon^r = \sum_{j=1}^{J_r} c_j^r \hat{W}_{11}^0(k_j^r), \qquad r = \mathrm{II}, \mathrm{IV}, \mathrm{V} \tag{7.41}$$

where $\{k_j^r\}_{j=1}^{J_r}$ is a discrete set of momenta in B, $c_j^r > 0$, for $j = 1, \ldots, J_r$, and

$$\sum c_j^r = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} \frac{1}{|\Lambda|} \sum_{i \in \Lambda} |q_i^r|^2$$

Suppose now that $\hat{q}(k)$ is the Fourier transform of a (periodic) function q on $\mathbb{Z}^2$. If $\hat{q}^2(k) = 0$ and $\hat{q}^1(k) \propto \delta[k - (0, \pm\pi)]$ then $q \propto q^{(\mathrm{I})}$. But $q^{(r)} \not\propto q^{(\mathrm{I})}$, $r = \mathrm{II}, \mathrm{IV}, \mathrm{V}$. Thus there exists j_0 such that $\operatorname{supp} \hat{q}^{(r)}(k) \ni k_{j_0}^r$, and $k_{j_0}^r \neq (0, \pm\pi)$, for $r = \mathrm{II}, \mathrm{IV}, \mathrm{V}$. By (7.41) and Lemma 7.4,

$$\epsilon^r \geqslant c_{j_0}^r \hat{W}_{11}^0(k_{j_0}^r)$$

and, by Proposition 7.5: the right-hand side is strictly positive.

Next, we note that

$$\hat{W}_{21}^0((0, k_2)) = \hat{W}_{12}^0((0, k_2)) = 0, \qquad -\pi \leqslant k_2 \leqslant \pi \tag{7.42}$$

We recall that the 1-component of $q^{(\mathrm{III})}$ is invariant under translation in the 1-direction. Thus, (7.42) implies

$$\frac{1}{|\Lambda|} \sum_{i,j \in \Lambda} \left(q_i^{(\mathrm{III})} \right)^1 W_{\Lambda,12}^0(i-j) \left(q_j^{(\mathrm{III})} \right)^2 = 0$$

This and translation invariance of W_Λ^0 now imply $\epsilon^{\mathrm{III}} = \epsilon^{\mathrm{IV}} + \epsilon^{\mathrm{V}}$ which is strictly positive by that we have already proven. ■

Corollary 7.6 is going to play an important role in the estimation of contour probabilities in the Peierls argument; see Section 7.5.

Next, we want to determine the set $S \subset B$ of momenta k_s such that $\hat{W}^0(k_s)$ is singular, i.e., has at least one zero eigenvalue. A momentum

$k_s \in S$ is called a singular momentum. Let $[\pi^{(\alpha)}]$ be the set given by $\pi^{(\alpha)}$ and all its $2^{\nu-1} - 1$ periodic images.

By periodicity, $\hat{W}^0(k) = \hat{W}^0(\pi^{(\alpha)})$, for all $k \in [\pi^{(\alpha)}]$. Since $\hat{W}^0_{\alpha\gamma}(\pi^{(\alpha)}) = \hat{W}^0_{\gamma\alpha}(\pi^{(\alpha)}) = 0$, $\gamma = 1, \ldots, \nu$, $\hat{W}^0(\gamma^{(\alpha)})$ has a zero eigenvalue, for all $\alpha = 1, \ldots, \nu$, i.e.,

$$\bigcup_{\alpha=1}^{\nu} \left[\pi^{(\alpha)}\right] \subseteq S \tag{7.43}$$

We now pose the problem to show that

$$\bigcup_{\alpha=1}^{\nu} \left[\pi^{(\alpha)}\right] = S \tag{7.44}$$

and to determine the behavior of $\hat{W}^0(k)$ for k in the vicinity of $[\pi^{(\alpha b)})$, for some α. We think that this problem can be solved for a very general class of dipole potentials obeying inequality (7.13) (i.e., reflection positivity). Unfortunately, the analysis in the general case appears to be rather subtle, and we therefore limit our considerations to a nearest-neighbor dipole potential.

Let $r = (x_1^2 + \cdots + x_\nu^2)^{1/2}$. For $\nu > 2$, the Coulomb potential is given by const $r^{-(\nu-2)}$. We set $C(x) = r^{-(\nu-2)}$, for $r \geqslant 1$. Consider the dipole potential

$$\begin{aligned} W^c_{\alpha\gamma}(x) &= -\frac{\partial^2}{\partial x^\alpha \partial x^\gamma} r^{-(\nu-2)} \\ &= (\nu-2)\frac{\delta_{\alpha\gamma}}{r^{\nu/2}} - \nu(\nu-2)\frac{x^\alpha x^\gamma}{r^{\nu/2+1}} \end{aligned} \tag{7.45}$$

The Coulomb potential is obviously a reflection positive two-point function $[d\mu(a) = \delta_0(a)da$ in (7.17)] and thus satisfies (7.15). Therefore the restriction of W^c to $\mathbb{Z}^\nu$, W, satisfies (7.13). The proof of the following lemma is trivial.

Lemma 7.7. Let W satisfy (7.13). Then W', defined by

$$W'(j) = \begin{Bmatrix} W(j), & |j| \leqslant 1 \\ 0, & |j| > 1 \end{Bmatrix}$$

satisfies (7.13). More generally, if $V(j)$ is an arbitrary reflection-positive two-point function then V', given by $V'(j) = V(j)$, for $|j| \leqslant 1$, $V'(j) = 0$, otherwise, is reflection positive, as well.

Let e_α be the unit lattice vector in the α direction, and p_{e_α} the orthogonal projection onto e_α. Then, for W^c as in (7.45),

$$\begin{aligned} &W'(j) = (\nu-2)1 - \nu(\nu-2)\sum_{\alpha=1}^{\nu} j^\alpha p_{e_\alpha} \qquad \text{for } |j| = 1, \\ &W'(0) \propto 1 \quad \text{and} \quad W'(j) = 0 \qquad \text{for } |j| > 1 \end{aligned} \tag{7.46}$$

By Lemma 7.7, W' satisfies (7.13).

The Fourier transform of W' is given by

$$\hat{W}'(k) = -\left[(\nu-2)\Delta(k)+c\right]1 - 2\nu(\nu-2)\sum_{\alpha=1}^{\nu} p_{e_\alpha}\cos k_\alpha \tag{7.47}$$

where $\Delta(k) = 2[\nu - \sum_{\alpha=1}^{\nu}\cos k_\alpha]$, c is some constant. Note that $\hat{W}'(k)$ is a diagonal matrix (in the obvious basis). We define

$$\hat{W}^0(k) = \hat{W}'(k) + \left[c + 2(\nu-2)(3\nu-2)\right]1 \tag{7.48}$$

Then $\hat{W}^0_{\alpha\alpha}(\pi^{(\alpha)}) = 0$, so that our normalization condition is satisfied.

The eigenvalues of $\hat{W}^0(k)$ are given by

$$\lambda_\alpha(k) = \hat{W}^0_{\alpha\alpha}(k) = (\nu-2)\left[6\nu - 4 - \Delta(k) - 2\nu\cos k_\alpha\right] \tag{7.49}$$

Equation (7.49) obviously shows that the only zeros of $\lambda_\alpha(k)$ are the points in $[\pi^{(\alpha)}]$. Moreover, $\lambda_\mu(\pi^{(\alpha)}) = 4\nu(\nu-2) > 0$, for $\mu \neq \alpha$. Finally,

$$\lambda_\alpha(k) \geqslant \delta(\nu-2)\,\mathrm{dist}\big(k,\left[\pi^{(\alpha)}\right]\big)^2, \qquad \delta \approx 1$$

for k near one of the points in $[\pi^{(\alpha)}]$. We summarize in the following lemma.

Lemma 7.8. For $\hat{W}^0$ as in (7.46)–(7.48),

$$S = \bigcup_{\alpha=1}^{\nu}\left[\pi^{(\alpha)}\right]$$

For $k \notin S$, $\hat{W}^0(k)^{-1}$ exists, and $[\hat{W}^0(k)^{-1}]_{\mu\mu} = \hat{W}^0_{\mu\mu}(k)^{-1}$ is bounded uniformly in the complement of any small, open neighborhood of $[\pi^{(\mu)}]$. Finally $0 < [\hat{W}^0(k)^{-1}]_{\alpha\alpha} = \hat{W}^0_{\alpha\alpha}(k)^{-1} \leqslant \{\delta(\nu-2)\,\mathrm{dist}(k,[\pi^{(\alpha)}])^2\}^{-1}$, $\delta \approx 1$, for k near $[\pi^{(\alpha)}]$.

For general dipole potentials, $W^c_{\alpha\gamma} = -\partial^2/\partial x^\alpha \partial x^\gamma C$, where C is given by (7.17) (with $\operatorname{supp} d\mu \subseteq \{0\} \cup [\epsilon,\infty)$, $\epsilon > 0$) one can show without major efforts that

$$\hat{W}^0(k) = U(k)\begin{bmatrix} \lambda_1(k) 0 & \cdots & 0 \\ 0 & \lambda_2(k)\cdots & 0 \\ 0 & \cdots & \lambda_\nu(k) \end{bmatrix} U(k)^*$$

for some unitary matrix with the property that

$$\begin{aligned} \lim_{k\to[\pi^{(\alpha)}]} U(k)_{\alpha\gamma} &= \delta_{\alpha\gamma} \\ \lim_{k\to[\pi^{(\alpha)}]} U(k)_{\gamma\alpha} &= \delta_{\alpha\gamma} \\ \lim_{k\to[\pi^{(\alpha)}]} \lambda_\alpha(k) &= 0 \end{aligned} \tag{7.50}$$

We make the following conjecture.

Conjecture 7.9. For the general class of dipole potentials discussed here,

$$S = \bigcup_{\alpha=1}^{\nu} [\pi^{(\alpha)}]$$

and

$$0 < \lambda_\alpha(k)^{-1} < \epsilon \operatorname{dist}\big(k, [\pi^{(\alpha)}]\big)^{-2} \tag{7.51}$$

for some finite ϵ.

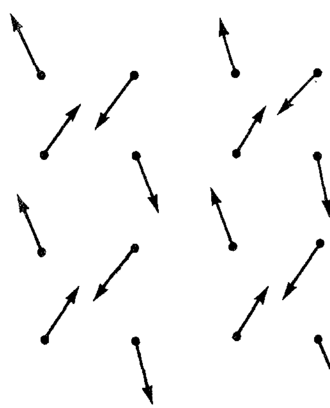
We remark that the gound-state configurations, q, of a dipole potential W^0 must satisfy

$$\operatorname{supp} \hat{q} \subseteq S$$

For the potential W^0 introduced in (7.46)–(7.48) or for some W^0 for which Conjecture 7.9 holds, $\operatorname{supp} \hat{q}^\alpha = [\pi^{(\alpha)}]$, if q is a ground-state configuration, i.e., q is given given by

$$q_j = R(j)q, \qquad q \in \mathbb{R}^\nu; \qquad \text{see (7.33)}$$

The following is a portrait of a two-dimensional ground-state configuration:



In Sections 7.3–7.5 we show that under suitable conditions the ordering persists at sufficiently small temperature and large activity.

7.3. Infrared Bounds for Dipole Two-Point Functions

We now return to the discussion of the dipole lattice gases specified in Eqs. (7.4), (7.6), and (7.7) of Section 7.1. We assume that the *a priori* distribution, $d\rho$, of the dipole moment satisfies

$$d\rho(R_\alpha q) = d\rho(q) \qquad \text{for all } \alpha; \qquad \text{see (7.12)}$$

Moreover, W is supposed to be reflection positive in the sense of inequality (7.13).

If we replace $d\rho$ by

$$d\rho_0(q) \equiv e^{(\beta/2)\Delta q^2} d\rho(q) \tag{7.52}$$

and W by $W^0 = W - \Delta 1$, with $\Delta = \Delta_{\pi^{(\alpha)}}$ defined in (7.26), (7.27), the Gibbs expectation $\langle \cdot \rangle$, see (7.7), remains unchanged. Moreover,

$$\begin{gathered} d\rho_0(R_\alpha q) = d\rho_0(q) \\ \hat{W}^0_{\alpha\alpha}(\pi^{(\alpha)}) = 0 \qquad \text{for all } \alpha \\ \sum_{i \in \Lambda} W^0(l-i)R(i) = 0 \end{gathered} \tag{7.53}$$

where $R(j) = \prod_{\alpha=1}^{\nu} R_\alpha^{j_\alpha}$; see (7.27), (7.28), and (7.32).

Proposition 7.2 (reflection positivity of $\langle \cdot \rangle_\Lambda$) permits one to derive the usual chessboard estimate, see Refs. 16, 14, and 6, which, by a general argument[14] yield the following theorem.

Theorem 7.10 (Gaussian Domination). Let h be an arbitrary $\mathbb{R}^\nu$-valued function on $\mathbb{Z}^\nu$. Then

$$\langle e^{-\beta(q, W^0 h)} \rangle \leqslant e^{(\beta^2/2)(h, W^0 h)} \tag{7.54}$$

Outline of Proof. We temporarily assume that $d\rho_0(q) = \rho_0(q) d^\nu q$, with $\rho_0 \in L^1(\mathbb{R}^\nu)$ and $\rho_0(q) > 0$, almost everywhere. We consider

$$Z_\Lambda(h) = \int e^{-\beta/2(q+h, W^0(p+h))} \prod_{i \in \Lambda} d\rho_0(q_i) \tag{7.55}$$

Define $F_h(q) = \rho_0(q-h)/\rho_0(q)$. By a change of variables $(q + h \to q)$ one gets

$$Z_\Lambda(h) = \int e^{-\beta/2(q, W^0 q)} \prod_{i \in \Lambda} F_{h_i}(q_i) d\rho_0(q_i)$$

Next notice that

$$\begin{aligned} (\theta_\alpha F_h)(q_j) = F_h(\theta_\alpha q_j) &= F_h(R_\alpha q_{r_\alpha j}) \\ &= F_{R_\alpha h}(q_{r_\alpha j}) \end{aligned} \tag{7.56}$$

since, by (7.53), $\rho_0(R_\alpha q - h) = \rho_0(q - R_\alpha h)$. The chessboard estimate[14,6] thus gives

$$\begin{aligned} &\int e^{-\beta/2(q, W^0 q)} \prod_{i \in \Lambda} F_{h_i}(q_i) d\rho_0(q_i) \\ &\leqslant \prod_{i \in \Lambda} \left[\int e^{-\beta/2(q, W^0 q)} \prod_{j \in \Lambda} F_{R(j-i)h_i}(q_j) d\rho_0(q_j) \right]^{1/|\Lambda|} \end{aligned} \tag{7.57}$$

By reversing the change of variables in all terms on the right-hand side of (7.57) $[q_j \to q_j + R(j-i)h_i]$, we obtain from (7.55)–(7.57)

$$Z_\Lambda(h) \leqslant \prod_{i \in \Lambda} Z_\Lambda(h^{(i)})^{1/|\Lambda|}$$

with $h_j^{(i)} = R(j-i)h_i$. The last identity in both (7.53) and (7.55) show that $Z_\Lambda(h^{(i)}) = Z_\Lambda(0) = Z_\Lambda$, so that

$$Z_\Lambda(h) \leqslant Z_\lambda$$

This is a rewriting of (7.54).

The case of an arbitrary measure $d\rho_0$ obeying (7.53) follows from the special case treated above by a limiting argument. ■

Next, we replace h by ϵh in (7.54), expand both sides to second order in ϵ, subtract 1, divide by ϵ^2, and take $\epsilon \downarrow 0$. This yields (using the normalization condition for W^0)

$$\langle |(q, W^0 h)|^2 \rangle \leqslant \beta^{-1}(h, W^0 h) \tag{7.58}$$

Let S be the set of singular momenta of $\hat{W}^0$ [i.e., $\hat{W}^0(k_s)$ has zero eigenvalue for $k_s \in S$; see Section 7.2]. Let h be of the form

$$h = (W^0)^{-1} g$$

with supp $\hat{g} \cap S = 0$. Then (7.58) yields

$$\langle |(q, g)|^2 \rangle \leqslant \beta^{-1}\left(g, (h^0)^{-1} g\right) \tag{7.59}$$

Upon Fourier transforming both sides of (7.59) one finds the following.

Corollary 7.11 (Infrared Bound)

$$0 \leqslant \hat{Q}(k) \leqslant \beta^{-1}\hat{W}^0(k)^{-1} \qquad \text{for } k \notin S \tag{7.60}$$

in the sense of an inequality between positive matrices. Here $\hat{Q}(k)$ is the matrix defined by

$$\hat{Q}(k)_{\alpha\gamma} = \langle \hat{q}^\alpha(k)\overline{\hat{q}^\gamma(k)} \rangle \tag{7.61}$$

It is standard to transfer (7.54) and (7.60) to the thermodynamic limit.

For the nearest neighbor potential W' defined in Section 7.2, (7.46)–(7.48), and, more generally, for any reflection-positive dipole potential for which Conjecture 7.9 holds, we can sharpen Corollary 7.11 in the following way: by Lemma 7.8 (or Conjecture 7.9), it is enough that the function g in inequality (7.59) has the property that

$$\hat{g}^\alpha(k) = 0 \qquad \text{for } k \in [\pi^{(\alpha)}] \tag{7.62}$$

Define a matrix-valued distribution, $M(k)$, by

$$M(k)_{\alpha\gamma} = \delta_{\alpha\gamma} m_\alpha \sum_{p \in [\pi^{(\alpha)}]} \delta(k-p) \tag{7.63}$$

with $m_\alpha \geqslant 0$, for all $\alpha = 1, \ldots, \nu$. Then inequality (7.59) and (7.62) yield, after Fourier transformation,

$$\boxed{0 \leqslant \hat{Q}(k) \leqslant \beta^{-1}\hat{W}^0(k)^{-1} + M(k)} \tag{7.64}$$

in the sense of inequalities between positive, matrix-valued distributions, in particular

$$0 \leqslant \langle |\hat{q}^\alpha(k)|^2 \rangle \leqslant \beta^{-1}\left[\hat{W}^0(k)^{-1} \right]_{\alpha\alpha} + m_\alpha \sum_{p \in [\pi^{(\alpha)}]} \delta(k-p) \tag{7.65}$$

for some $m_\alpha \geqslant 0$.

Fourier transformation of (7.65) shows that

$$\langle q_0^\alpha q_j^\alpha \rangle \approx m_\alpha (-1)^{\sum_{\mu \neq \alpha} j^\mu} \tag{7.66}$$

as $|j| \to \infty$, i.e., $\langle \cdot \rangle$ is *not* an extremal Gibbs state if $m_\alpha > 0$, for some $\alpha = 1, \ldots, \nu$. In the last two sections we show that, under suitable assumptions on W and $d\rho$, $\sum_{\alpha=1}^{\nu} m_\alpha > 0$, for β sufficiently large. We close by noticing that if $\langle \cdot \rangle$ is the thermodynamic limit of states $\langle \cdot \rangle_\Lambda$ which are symmetric under exchanging coordinate axes (i.e., Λ is a hypercube), then $m_1 = \cdots = m_\nu$, and $\langle \cdot \rangle$ is a mixture of at least 2ν extremal Gibbs states, $\langle \cdot \rangle^{(\lambda)}$, which break translation invariance and are characterized by

$$\langle q_j \rangle^{(\lambda)} = R(j) q^{(\lambda)} \tag{7.67}$$

where $\{q^{(\lambda)} : \lambda = 1, \ldots, 2\nu, \ldots\}$ are vectors obtained from some vector $q^{(1)} \in \mathbb{R}^\nu$ by applying arbitrary rotations around the origin which leave the unit cube centered at the origin invariant. This follows from the assumed symmetry of $\langle \cdot \rangle$, by the general theory of decomposition into extremal states.

7.4. Lower Bounds on $\langle q_0^2 \rangle$

In this section we establish uniform lower bounds on $\langle q_0^2 \rangle$ which we then exploit in conjunction with the basic infrared bound (7.64) of Section 7.3 (and uniqueness for small β or small activity) to complete the proof of existence of phase transitions at small temperatures, as the activity is varied, in $\nu \geqslant 3$ dimensions.

We follow the standard strategy.[20] Consider the nearest-neighbor dipole potential W^0 defined by

$$W^0(j) = \begin{cases} (\nu-2)1 - \nu(\nu-2) \sum_{\alpha=1}^{\nu} j^\alpha p_{e_\alpha} & \text{for } |j| = 1 \\ 2(\nu-2)(3\nu-2)1 & \text{for } |j| = 0 \\ 0 & \text{otherwise} \end{cases} \tag{7.68}$$

where p_{e_α} is the orthogonal projection onto a unit lattice vector, e_α; see (7.46)–(7.48), Section 7.2. By the basic infrared bound, inequality (7.64) of Section 7.3,

$$0 \leqslant \langle |\hat{q}(k)|^2 \rangle = \mathrm{Tr}\hat{Q}(k) \leqslant \beta^{-1}\mathrm{Tr}\hat{W}^0(k)^{-1} + \mathrm{Tr}M(k) \tag{7.69}$$

with

$$M(k)_{\alpha\gamma} = \delta_{\alpha\gamma} m_\alpha \sum_{p \in [\pi^{(\alpha)}]} \delta(k-p)$$

Integrating both sides of (7.69) in k over the first Brillouin zone yields

$$0 < \langle q_0^2 \rangle \leqslant \beta^{-1}\int_B d^\nu k \,\mathrm{Tr}\, \hat{W}^0(k)^{-1} + \sum_{\alpha=1}^{\nu} m_\alpha \tag{7.70}$$

For the potential W^0 specified in (7.68), we may apply Lemma 7.8, Section 7.2 to estimate

$$I(\nu, W^0) \equiv \int_B d^\nu k \,\mathrm{Tr}\, \hat{W}^0(k)^{-1} \tag{7.71}$$

That lemma shows that

$$I(\nu, W^0) \text{ is finite for } \nu \geqslant 3 \tag{7.72}$$

[We note that (7.69), (7.70), and (7.72) also hold for each dipole potential for which Conjecture 7.9 is true.] Thus

$$0 < \langle q_0^2 \rangle \leqslant \beta^{-1} I(\nu, W^0) + M \qquad \text{with } M = \sum_{\alpha=1}^{\nu} m_\alpha \tag{7.73}$$

When $\nu \geqslant 3$ it suffices to prove, e.g., a uniform (in β) lower bound on $\langle q_0^2 \rangle$ in order to show that the long-range order M is strictly positive. Let χ_δ be the characteristic function of $\{q \in \mathbb{R}^\nu : |q| \leqslant \delta\}$. By the chessboard estimate and the fact that $\chi_\delta(q) = \chi_\delta(R_\alpha q)$, for all α, we have

$$\begin{aligned} \langle \chi_\delta(q_0) \rangle_\Lambda &= \left\langle \prod_{j \in \Lambda} \chi_\delta(q_j) \right\rangle_\Lambda^{1/|\Lambda|} \\ &= \left(Z_\Lambda^{-1} \int e^{-(\beta/2)(q, W_\Lambda q)} \prod_{j \in \Lambda} \chi_\delta(q_j) d\rho(q_j) \right)^{1/|\Lambda|} \\ &\equiv m_\Lambda(\delta, \beta) \end{aligned} \tag{7.74}$$

Suppose now that

$$\varliminf_{\Lambda \nearrow \mathbb{Z}^\nu} m_\Lambda(\delta, \beta) \equiv m(\delta, \beta) < 1 \tag{7.75}$$

From (7.74) and (7.75) we get

$$\langle q_0^2 \rangle \geqslant \delta^2[1 - m(\delta, \beta)] > 0 \tag{7.76}$$

In order to derive (7.75) we use the following estimates:

$$
\begin{aligned}
H_\Lambda &\equiv (\beta/2)(q, W_\Lambda q) = \sum_k (\hat{q}(k), \hat{W}_\Lambda(k)\hat{q}(k)) \\
&\leqslant \sup_k \|\hat{W}_\Lambda(k)\| \sum_k |\hat{q}(k)|^2 \\
&= \|W_\Lambda\| \sum_{i\in\Lambda} |q_i|^2
\end{aligned}
$$

Similarly,

$$
H_\Lambda \geqslant -\|W_\Lambda\| \sum_{i\in\Lambda} |q_i|^2
$$

Thus

$$
\begin{aligned}
&\left[\int e^{-H_\Lambda} \prod_{j\in\Lambda} \chi_\delta(q_j)\, d\rho(q_j) \right]^{1/|\Lambda|} \\
&\qquad \leqslant \int \chi_\delta(q) e^{(\beta/2)\|W_\Lambda\|\,|q|^2} d\rho(q)
\end{aligned}
\tag{7.77}
$$

and

$$
Z^{1/|\Lambda|} \geqslant \int e^{-(\beta/2)\|W\|\,|q|^2} d\rho(q) \tag{7.78}
$$

Now, for all dipole potentials considered in this paper,

$$
\overline{\lim_{\Lambda\uparrow\mathbb{Z}^\nu}} \|W_\Lambda\| \equiv \|W\| < \infty
$$

Thus

$$
\begin{aligned}
m(\delta, \beta) &\leqslant \left[\int \chi_\delta(q) e^{(\beta/2)\|W\|\,|q|^2} d\rho(q) \right] \\
&\qquad \times \left[\int e^{-(\beta/2)\|W\|\,|q|^2} d\rho(q) \right]^{-1}
\end{aligned}
\tag{7.79}
$$

For the potential W^0 defined in (7.68) this estimate can be improved: since $\hat{W}^0(k) \geqslant 0$ (see Lemma 7.4),

$$
m(\delta, \beta) \leqslant \left[\int \chi_\delta(q)\, d\rho(q) \right] \left[\int e^{-(\beta/2)\|W^0\|\,|q|^2} d\rho(q) \right]^{-1} \tag{7.80}
$$

and from Proposition 7.5, (2) and the fact that $\hat{W}^0$ is diagonal,

$$
\|W^0\| = \hat{W}^0_{11}[(\pi, 0, \ldots, 0)]
$$

so that by (7.68), and (7.47), (7.48),

$$
\|W^0\| = 8(\nu - 2)(\nu - 1) \tag{7.81}
$$

Suitable hypotheses on $d\rho$ together with (7.73), (7.76), (7.80), and (7.81) suffice to show that

$$M \geqslant \sup_{\delta} \delta^2\left[1 - m(\delta, \beta)\right] - \beta^{-1}I(\nu, W^0) > 0$$

in the appropriate range of β's.

We now consider an example of a distribution $d\rho$ corresponding to a (Dhc) ensemble:

$$d\rho(q) = \left[\delta_0(q) + z\delta(|q| - Q)\right]d^\nu q \tag{7.82}$$

Choosing $\delta < Q$ we have

$$\int \chi_\delta(q)\,d\rho(q) = 1$$

and

$$\int e^{-(\beta/2)\|W^0\|\,|q|^2}\,d\rho(q) \geqslant 1 + ze^{-\beta\mu} \tag{7.83}$$

where

$$\mu \equiv \mu(\nu, Q) = 4(\nu - 2)(\nu - 1)\,Q^2$$

Thus

$$M \geqslant Q^2 ze^{-\beta\mu}\left(1 + ze^{-\beta\mu}\right)^{-1} - \beta^{-1}I(\nu, W^0) \tag{7.84}$$

Thus, for $z = z_0 e^{\beta\mu}$,

$$M \geqslant Q^2 z_0(1 + z_0)^{-1} - \beta^{-1}I(\nu, W^0) > 0 \tag{7.85}$$

for

$$\beta > I(\nu, W^0)(1 + z_0)z_0^{-1}Q^{-2}$$

Equivalently, if $\beta > I(\nu, W^0)Q^{-2}$ then

$$M > 0 \qquad \text{for } z_0 > \left[\,Q^2\beta I(\nu, W^0)^{-1} - 1\right]^{-1} \tag{7.86}$$

Since $M = \sum_{\alpha=1}^{\nu} m_\alpha$, with $m_\alpha \geqslant 0$, for all α, $M > 0$ implies that $m_\alpha > 0$, for at least one α. As remarked at the end of Section 7.3, there then exist at least 2ν extremal Gibbs states, $\langle\,\cdot\,\rangle^{(\lambda)}$, with

$$\langle q_j\rangle^{(\lambda)} = R(j)q^{(\lambda)}$$

for some nonzero vectors $q^{(\lambda)} \in \mathbb{R}^\nu$, $\lambda = 1, \ldots, 2\nu, \ldots$ related to each other by sequences of 90° rotations. We remark that the results proven in this and the last section for the nearest-neighbor dipole potential W^0 defined in (7.68) can be extended to all dipole potentials for which Conjecture 7.9 can be proven.

Finally, we point out that for the nearest-neighbor dipole potential W^0 defined in (7.68) or (7.46)–(7.48) a standard high-temperature expansion yields uniqueness and exponential clustering of $\langle \cdot \rangle$ at small values of β, for all z. If W is a lattice dipole potential of arbitrary range chosen such that $\hat{W}(k)$ is invertible for all $k \in B$ (see Section 7.2) we can use inequality (2.35), Section 2 to show that for $z < 1$

$$\hat{Q}(k) \leqslant [\beta \hat{W}(k)]^{-1}$$

where

$$\hat{Q}(k)_{\alpha\gamma} = \langle \hat{q}^{\alpha}(k) \overline{\hat{q}^{\gamma}(k)} \rangle$$

This inequality (with the Riemann–Lebesque lemma) proves absence of long-range order in the two-point function, $\langle q_0 q_j \rangle$, for all β and all $z < 1$. (The techniques of Section 5 may permit extending this to all $z < e^{0(\beta)}$.)

7.5. Two-Dimensional Dipole Gases: The Peierls Argument

In two dimensions the techniques of Sections 7.3 and 7.4 are certainly not applicable. However, for a very general class of dipole potentials satisfying reflection positivity [see (7.13) and (7.15) in Section 7.1] and discrete distribution $d\rho$, e.g.,

$$d\rho(q) = \{\delta_0(q) + z_1[\delta(q - Qe_1) + \delta(q + Qe_1)] + z_2[\delta(q - Qe_2) + \delta(q + Qe_2)]\} d^2q \tag{7.87}$$

one can use the Peierls chessboard method (see Refs. 5, 14, and 6) to establish ordering for sufficiently large β, z_1, and z_2. This method can be used in arbitrary dimension $\nu \geqslant 2$, as long as the symmetry group leaving $d\rho$ invariant is discrete. This is of considerable interest as long as Conjecture 7.9 is unproven for long-range dipole potentials. We briefly sketch the method for $\nu = 2$ and $d\rho$ as in (7.87). Details and generalizations to $\nu > 2$ and a large class of $d\rho$ are straightforward and can be inferred from the references quoted above. Let

$$\chi_{\substack{u\\d}} = \begin{cases} 1, & q = \pm Qe_1 \qquad (\text{i.e., } q = \uparrow \text{ or } \downarrow) \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_{\substack{r\\l}} = \begin{cases} 1, & q = \pm Qe_2 \qquad (\text{i.e., } q = \rightarrow \textit{ or } \leftarrow) \\ 0 & \text{otherwise} \end{cases}$$

$$\chi_0 = \begin{cases} 1, & q = 0 \qquad (\text{i.e., } q = \emptyset) \\ 0 & \text{otherwise} \end{cases}$$

Where convenient we identify $\{0, u, d, r, l\}$ with $\{0, 1, 2, 3, 4\}$. We define

$$P_s(j) = \chi_s[R(j)q_j]$$

with $R(j) = \prod_{\alpha=1}^{\nu} R_\alpha^{j^\alpha}$, and R_α as in (7.8), Section 7.1. One then checks, using (7.9) and (7.10), of Section 7.1,

$$\theta_\alpha P_s(j) = P_s(r_\alpha j) \tag{7.88}$$

for all s and all $\alpha = 1, \ldots, \nu$. For $d\rho$ given by (7.87)

$$P_0(j) + P_u(j) + P_d(j) + P_r(j) + P_l(j) = 1 \tag{7.89}$$

for all $j \in \mathbb{Z}^2$.

We choose W to be a general, reflection-positive dipole potential of the type studied in Section 7.2 for which, in particular, Corollary 7.6 of Section 7.2 is valid. We then choose $\langle \cdot \rangle$ to be a some limit of a sequence of periodic states, $\langle \cdot \rangle_\Lambda$. These states (and thus any limit) satisfy reflection positivity (see Propositions 7.2 and 7.3 of Section 7.1), permitting the application of the chessboard estimates,[14] and are symmetric under exchanging u with d and r with l. Thus we have

$$\langle P_u(j)\rangle = \langle P_d(j)\rangle, \qquad \langle P_r(j)\rangle = \langle P_l(j)\rangle \qquad \text{for all } j$$

Furthermore

$$\langle P_u(j)\rangle \geqslant \langle P_r(j)\rangle \qquad \text{for } z_1 \geqslant z_2$$

Thus

$$\langle P_u(j)\rangle = \langle P_d(j)\rangle > \tfrac{1}{4}(1 - \langle P_0\rangle) \tag{7.90}$$

In order to prove that $\langle \cdot \rangle$ violates clustering (i.e., is not extremal), we propose to show that

$$\langle P_u(0)P_d(j)\rangle \leqslant e^{-K\beta} \qquad \text{for all } j \tag{7.91}$$

and

$$\langle P_0\rangle \leqslant e^{-K'\beta} \tag{7.92}$$

for some positive constants K and K' and all β. Obviously, (7.90)–(7.92) prove that for β large enough $\langle P_u(0)P_d(j)\rangle \nrightarrow \langle P_u\rangle\langle P_d\rangle$, as $|j| \to \infty$. By (7.88) and the chessboard estimate,

$$\langle P_0\rangle \leqslant \lim_{\Lambda\uparrow\mathbb{Z}^2} \left\langle \prod_{j\in\Lambda} P_o(j) \right\rangle_\Lambda^{1/|\Lambda|} = \lim_{\Lambda\uparrow\mathbb{Z}^2} (Z_\Lambda)^{-1/|\Lambda|}$$

Now $Z_\Lambda > z_1 e^{-\beta Q^2 \epsilon^{\mathrm{I}} |\Lambda|}$, where ϵ^{I} is the ground state energy defined in (7.40), Section 7.2. We choose z_1 such that

$$z_1 e^{-\beta\epsilon^{\mathrm{I}} Q^2} = e^{K'\beta}, \quad \text{i.e.,} \quad \beta\mu_1 \equiv \log z_1 = \beta(K' + \epsilon^{\mathrm{I}} Q^2) \tag{7.93}$$

This yields (7.92).

To prove (7.91) we apply the standard Peierls argument: by (7.89),

$$\langle P_u(0)P_d(j)\rangle = \left\langle P_u(0)P_d(j) \prod_{i\in\Omega\setminus\{0,j\}} \left[\sum_{\alpha=0}^{4} P_\alpha(i) \right] \right\rangle$$

where Ω is an arbitrary, bounded square in $\mathbb{Z}^2$, containing 0 and j. The right-hand side is then expanded and resummed, using $0 \leqslant P_\alpha(j) \leqslant 1$, for all α and j. This yields, after taking $\Omega \uparrow \mathbb{Z}^2$,

$$\langle P_u(0) P_d(j) \rangle \leqslant \sum_\gamma \Big\langle \prod_{(i,i') \in \gamma} P_\alpha(i) P_{\alpha'}(i') \Big\rangle$$

where γ labels an arbitrary contour $\gamma \subset \mathbb{Z}^2$ consisting of finitely many pairs of nearest neighbors (i, i') separating 0 from j. (See Refs. 5 and 14 for precise definitions.) Here

$$\alpha = \alpha' = 0 \quad \text{or} \quad \alpha \neq \alpha' \qquad \text{for all } (i, i') \tag{7.94}$$

Applying the chessboard estimate to the right-hand side as in Ref. 14, using (7.88) we obtain the upper bound

$$\langle P_u(0) P_d(j) \rangle \leqslant \sum_\gamma \max_{\alpha, \alpha'} \lim_{\Lambda \uparrow \mathbb{Z}^2} \Big\langle \prod_{(i,i') \subset \Lambda} P_\alpha(i) P_{\alpha'}(i') \Big\rangle_\Lambda^{|\gamma|/2|\Lambda|} \tag{7.95}$$

where α and α' are as in (7.94), the product on the right-hand side of (7.95) extends over all "horizontal" nearest-neighbor pairs, with α, α' depending only on $(-1)^{i^2}$, and $|\gamma|$ is the number of *pairs* in γ. Each term under the sum on the right-hand side of (7.95) is a thermodynamic quantity that can be estimated explicitly. One sees by inspection that

$$\lim_{\Lambda \uparrow \mathbb{Z}^2} \Big\langle \prod_{(i,i') \subset \Lambda} P_\alpha(i) P_{\alpha'}(i') \Big\rangle_\Lambda^{|\gamma|/2|\Lambda|} \leqslant (\Gamma_{\alpha\alpha'})^{|\gamma|} \tag{7.96}$$

with $\Gamma_{00} = z_1^{-1/2}$, and

$$\Gamma_{\alpha\alpha'} \leqslant \exp\big[-\beta/4(\epsilon^{r(\alpha,\alpha')} - \epsilon^{I}) Q^2 \big] \qquad \text{for } \alpha \neq \alpha'$$

where the energy densities ϵ^r, $r = \mathrm{I}, \ldots, \mathrm{VI}$ are defined in (I)–(VI) and (7.40) of Section 7.2. When $\alpha \neq \alpha'$, $r(\alpha, \alpha') \in \{\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}\}$. By Corollary 7.6 of Section 7.2,

$$\epsilon^r - \epsilon^{\mathrm{I}} > 0 \qquad \text{for } r = \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{V}$$

Thus

$$\max_{\alpha,\alpha'} \Gamma_{\alpha\alpha'} \leqslant e^{-K''\beta} \tag{7.97}$$

for some constant $K'' > 0$, and α, α' as in (7.94). The main inequality (7.91) follows from (7.95)–(7.97), by the usual combinatorial arguments; see, e.g., Refs. 5, 14.

This completes the proof that $\langle \cdot \rangle$ violates clustering for large β.

We emphasize that the range of the dipole potential W is arbitrary, and that only the discrete nature of $d\rho$ [*not* the explicit choice (7.87)] was important, throughout Section 7.5.

APPENDIX A. REFLECTION POSITIVITY (RP)

This appendix briefly reviews reflection positivity for monopole and dipole gases in both the ϕ and q representations. We assume the reader has some familiarity with reflection positivity as developed in Refs. 13 and 14.

Let L_0 be the hyperplane $x^0 = 1/2$ lying between the points of the lattice $\mathbb{L} = \mathbb{Z}^\nu$ and define r to be reflection through L_0. We set

$$\mathbb{L}_+ = \{x \in \mathbb{L} : x^0 \geqslant 1\}$$

$$\mathbb{L}_- = \{x \in \mathbb{L} : x^0 \leqslant 0\}$$

and denote by $\mathbb{F}_\pm$ functions of $\{\phi(x)\}_{x\in\mathbb{L}_\pm}$.

Definition A.1. A quadratic form C on $l_2(\mathbb{L})$ is called reflection positive (RP) if $C(x,y) = C(rx,ry)$ and

$$\langle \theta f, Cf\rangle_{l_2(\mathbb{L})} \geqslant 0, \qquad \operatorname{supp} f \subset \mathbb{L}_+$$

where

$$(\theta f)(x) = f(rx)$$

By general properties of Gaussian measures[15] we have the following proposition.

Proposition A.1. If C is RP then for $A \in \mathbb{F}_+$

$$\langle \overline{\theta A}\, A\rangle_{\beta C} \geqslant 0$$

where

$$(\theta A)\{\phi(x)\} = A\{\phi(rx)\}$$

Remark. For $C = \beta(-\Delta + \epsilon)^{-1}$, Proposition A.1 follows from the fact that $\exp[-(1/2\beta)(\phi - \phi')^2]$ is the kernel of a positive operator corresponding to the transfer matrix.

Corollary A.2. For the monopole gas ensemble (Mg) with F satisfying the neutrality condition

$$d\lambda(q) = d\lambda(-q) \quad \text{and} \quad \Lambda = r\Lambda$$

we have

$$\langle \overline{\theta A}\, A\rangle_\Lambda(\beta; F) \geqslant 0$$

Proof. Let $F_\Lambda = \prod_{j\in\Lambda} F[\phi(j)]$ and let $\Lambda_\pm = \Lambda \cap \mathbb{L}_\pm$. Since F is real and $F_\Lambda = F_{\Lambda_+} F_{\Lambda_-}$, with $F_{\Lambda_-} = \theta(F_{\Lambda_+})$,

$$\langle \overline{\theta A}\, A\rangle_\Lambda(\beta; F) = \langle F_\Lambda\rangle^{-1}\big\langle \theta\big(\overline{AF}_{\Lambda_+}\big)\big(AF_{\Lambda_+}\big)\big\rangle_{\beta C}$$

By Proposition A.1 both factors on the right are positive. ■

Remark. Suppose that the limiting state,

$$\langle\cdot\rangle(\beta; F) = \lim_{\substack{\Lambda\uparrow\mathbb{L} \\ \Lambda = r\Lambda}} \langle\cdot\rangle_\Lambda(\beta; F)$$

is translation invariant (see the remarks after Theorem 2.4). Then it admits a positive semidefinite transfer matrix, T_ϕ. See Refs. 13 and 14.

Define a scalar product on the space, $\mathbb{F}_0$, of functions $A, B, \ldots$ of $\{\phi(x): x = (x^1, \ldots, x^\nu) \in \mathbb{L}, x^1 = 0\}$ by

$$\langle A, B\rangle = \langle \bar{A}B\rangle(\beta; F)$$

Let B_x be the translate of B by the vector $x \in \mathbb{L}$. Then for A, B in $\mathbb{F}_0$

$$\langle \bar{A}B_x\rangle(\beta; F) = \langle A, (T_\phi)^{|x^1|}B_{(0, x^2, \ldots, x^\nu)}\rangle \tag{A.1}$$

Let e be a unit vector in an arbitrary direction of $\mathbb{L}$, e.g., the x^1 direction, and let $A \in \mathbb{F}_0$. It then follows from (A.1) and the positivity of T_ϕ that

$$\langle \bar{A}A_{ne}\rangle(\beta; F) = \langle \bar{A}A_{|n|e}\rangle(\beta; F) \tag{A.2}$$

and

$$\langle \bar{A}A_{ne}\rangle(\beta; F) \text{ is convex on } n = 0, 1, 2, \ldots \tag{A.3}$$

This is applied in Section 4.

Next, we reformulate Corollary A.2 in the q representation. Let $q_\pm = \{q_x : x \in \mathbb{L}_\pm \cap \Lambda\}$, and define

$$(\theta q)_x = -q_{rx}$$
$$(\theta A)\{q_x\} = A\{-q_{rx}\} \tag{A.4}$$

Corollary A.3. Consider the monopole gas ensemble. Assume that C is RP, $d\lambda(q) = d\lambda(-q)$ and $\Lambda = r\Lambda$. For an arbitrary function $A(q_+)$ we have

$$\langle \overline{\theta A}\, A\rangle_\Lambda(\beta; F) \geqslant 0$$

Proof. This may be seen by applying Corollary A.2 to the function

$$A(\phi_+) \equiv \int dP(q_+) \prod_{x\in\mathbb{L}_+} e^{iq_x\phi(x)} \in \mathbb{F}_+$$

where dP is an arbitrary, complex measure on $\mathbb{R}^{|\mathsf{L}_+|}$. The sign change in θq [right-hand side of (A.4)] comes from complex conjugation.

By general arguments[13,14] there exists then a self-adjoint transfer matrix, T_q. One can deduce from (A.3), (A.4), and (2.29) that T_q is *not* positive, in contrast to the transfer matrix, T_ϕ, of the monopole gas in the ϕ representation. See Section 4.

We now turn to the discussion of *RP* for the *dipole gas* in the (Dg) ensemble.

Let ϕ be the Gaussian process over $\mathbb{R}^\nu$ with mean 0 and covariance βC. Let $\mathsf{L} = L\mathbb{Z}^\nu$ be the simple cubic lattice of mesh $L(=1,2,3,\ldots)$.

$$\left.\begin{array}{l}\text{We define } \mathbb{R}^\nu_\pm = \{x \in \mathbb{R}^\nu : x^0 \gtreqless 0\}. \text{ Let } \mathsf{O}_0 \text{ be a finite, closed set of}\\ \text{points contained in a square with sides of length } < L \text{ parallel to the axes}\\ \text{of } \mathsf{L}, \text{ centered at the origin } O \in \mathsf{L}.\end{array}\right\} \tag{A.5}$$

The translate of O_0 to a site $x \in \mathsf{L}$ is denoted O_x. We define

$$\begin{aligned} \mathsf{O}_\pm &= \bigcup_{x \in \mathsf{L}_\pm} \mathsf{O}_x \subseteq \mathbb{R}^\nu_\pm \\ \mathsf{S}_{\mathsf{O}_\pm} &= \{f : \operatorname{supp} f \subseteq \mathsf{O}_\pm\} \end{aligned} \tag{A.6}$$

and

$$\mathsf{F}_{\mathsf{O}_{(\pm)}} \text{ the functions of } \{\phi(x)\}_{x \in \mathsf{O}_{(\pm)}}$$

Definition A.2. A quadratic form C on $L^2(\mathbb{R}^\nu)$ is said to be O_+-RP iff $C(x,y) = C(rx, ry)$, for all x, y in O_+, and

$$(\phi f, Cf) \geqslant 0 \qquad \text{for all } \mathsf{F} \in \mathsf{S}_{\mathsf{O}_+} \tag{A.7}$$

Examples

(1) Clearly $C = (-\Delta + \epsilon)^{-1}$ is O_+-RP.

(2) Let $d = \operatorname{dist}(\mathsf{O}_+, \mathsf{O}_-)$. By (A.5), $d > 0$. Define $C'(x-y)$ by

$$C'(x-y) = \begin{cases} \text{integral kernel of } (-\Delta+\epsilon)^{-1} & \text{if } |x-y| \geqslant d \\ g(x-y) & \text{if } |x-y| < d \end{cases} \tag{A.8}$$

for an arbitrary function g.

Then C' is O_+-RP. The *proof* is as follows:

$$(\theta f, C'f) = (\theta f, Cf) \qquad \text{for all } f \in \mathsf{S}_{\mathsf{O}_+}$$

because $C'(x-y) = C(x-y)$ when $|x-y| \geqslant d$, and $\operatorname{dist}(\operatorname{supp} f, \operatorname{supp} \theta f) \geqslant \operatorname{dist}(\mathsf{O}_+, \mathsf{O}_-) = d$.

(3) Let $C = (-\Delta + \epsilon)^{-1}$, where Δ is the finite difference Laplacean on $l_2(\mathbb{Z}^\nu)$, and let $\mathcal{O}_0$ be an arbitrary subset of sites in $\mathbb{Z}^\nu$ of distance $\leqslant (L/2)$ to 0. Then C is $\mathcal{O}_+$-RP.

(A general way of constructing $\mathcal{O}_+$-RP C's can be inferred from Ref. 14.)

We now recall the definition (2.25), (2.26) of the (Dg) ensemble: we choose F of the form

$$F(\phi_x) = \int d\lambda(q) e^{i(\delta_q \phi)(x)} \in \mathcal{F}_{\mathcal{O}_x} \tag{A.9}$$

see (A.6).

For δ_p as in (2.24), (A.9) holds if

$$\operatorname{supp} d\lambda \subseteq \mathcal{O}_0, \quad \text{e.g.,} \quad \operatorname{supp} d\lambda \subseteq \{ q : |q| \leqslant L/2 \}. \tag{A.10}$$

In accordance with (2.24), (2.25), and Proposition A.1 we define

$$\overline{\theta\left(e^{i(\delta_q \phi)(x)}\right)} = e^{i(\delta_{Rq}\phi)(rx)} \tag{A.11}$$

where, for $q = (q^0, q^1, \ldots, q^{\nu-1})$,

$$Rq = (q^0, -q^1, \ldots, -q^{\nu-1}) \tag{A.12}$$

Moreover

$$(\theta q)_x = Rq_{rx} \tag{A.13}$$

That this definition of the reflection of dipole moments is the right one can be understood by viewing a dipole as two oppositely charged monopoles and then applying (A.4) (Fig. 1).

We now suppose that

$$d\lambda(q) = d\lambda(Rq) \tag{A.14}$$

which is again some sort of neutrality condition. Assuming (A.9), (A.11), and (A.14) we find

$$(\theta F)(\phi_x) = F(\phi_{rx}) \tag{A.15}$$

for all $x \in \mathcal{L}_+$.

Using Proposition A.1 and (A.15) we conclude the following.

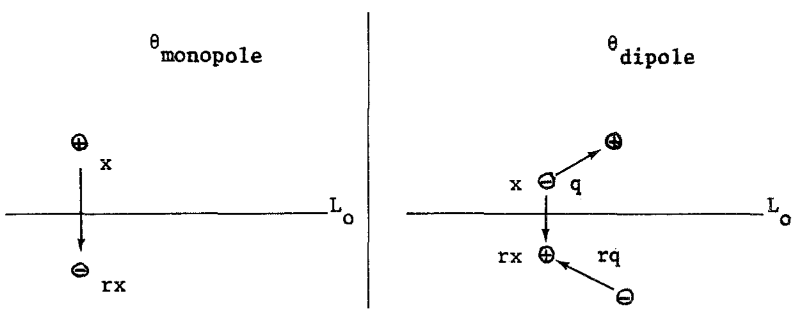


Fig. 1

Corollary A.4. Assume (A.9), (A.11), and (A.14). Suppose that $\Lambda = r\Lambda$ is reflection invariant. Then for all $A \in \mathsf{F}_{\mathsf{O}_+}$

$$\langle \overline{\phi A}\, A\rangle_\Lambda(\beta; F) \geqslant 0$$

Remark. We note that Corollary A.4 remains true for the (Dnhc) and the (Dhc) ensembles with $\mathsf{L} = \mathbb{Z}^\nu$, $\mathbb{O}_+ = \mathsf{L}_+$, and $(\delta_q\phi)(x) = \phi(x+q) - \phi(x)$, where q is an arbitrary lattice unit vector, and $d\lambda$ obeys (A.14). The proof follows from the fact that $\exp[z\cos(\phi - \phi')]$ or $1 + z\cos(\phi - \phi')$, $z > 0$, are obviously the integral kernels of *positive* quadratic forms (the Fourier transforms of $\exp z\cos\phi$ and $1 + z\cos\phi$ are nonnegative), by the arguments used in Ref. 16.

Next, let A be a function of q_+, where

$$q_\pm = \{q_x \in \operatorname{supp} d\lambda\}_{x \in \mathsf{L}_\pm} \tag{A.16}$$

We define

$$(\theta A)(q_-) = \overline{A(\theta q_-)} \tag{A.17}$$

Mimicking the arguments used to prove Corollary A.3—*mutatis mutandis*—and (A.13), (A.17) we get

Corollary A.5. Under the hypotheses of Corollary A.4,

$$\langle \overline{\theta A} \cdot A\rangle_\Lambda(\beta; F) \geqslant 0$$

for arbitrary functions A *only* depending on q_+.

Further discussion and important applications of Corollary A.5 (infrared bounds and existence of phase transitions) can be found in Section 7.

APPENDIX B. COMPLEX TRANSLATIONS AND ELECTROSTATICS

Let Σ be a connected, bounded region in $\mathbb{Z}^\nu$ and ρ some charge density inside Σ such that

$$\operatorname{dist}(\operatorname{supp}\rho, \partial\Sigma) > 0$$

Let

$$C_\rho(x) = \sum C(x-y)\rho(y) \tag{B.1}$$

Notice that $C\rho$ is *linear* in ρ.

We now look for a charge density $\sigma \equiv \sigma_\rho$ on $\partial\Sigma$ with the property that

$$C_\rho(x) = C_\sigma(x) \qquad \text{for } x \in \partial\Sigma \cup \Sigma^c \tag{B.2}$$

with $\partial\Sigma$ the boundary and Σ^c the complement of Σ. If (B.2) holds then, by linearity,

$$C_\rho(x) - C_\sigma(x) = C_{\rho-\sigma}(x) = 0 \qquad \text{for } x \in \partial\Sigma \cup \Sigma^c \tag{B.3}$$

Moreover,

$$(-\Delta C_{\rho-\sigma})(x) = \rho(x) \qquad \text{for } x \notin \partial\Sigma \tag{B.4}$$

since $\sigma(x) = 0$, for $x \notin \partial\Sigma$.

Thus $C_{\rho-\sigma}$ is the potential created by ρ with 0-Dirichlet data at $\partial\Sigma$, i.e.,

$$C_{\rho-\sigma}(x) \equiv C_\rho^D(x) = \sum_{y\in\Sigma} C^D(x,y)\rho(y) \tag{B.5}$$

where C^D is the Green's function of $-\Delta$ with 0-Dirichlet data at $\partial\Sigma$. By applying the Laplacean with free boundary conditions to both sides of this equation we find

$$(-\Delta C_{\rho-\sigma})(x) = -\sigma(x) = \left(-\Delta C_\rho^D\right)(x), \qquad x \in \partial\Sigma$$

i.e.,

$$\sigma(x) = \left(\Delta C_\rho^D\right)(x) \quad \text{for } x \in \partial\Sigma \tag{B.6}$$

Next, notice that

$$C_\rho^D(x) \geqslant 0 \qquad \text{if } \rho \geqslant 0 \tag{B.7}$$

because $C^D(x, y) \geqslant 0$ and $\rho(y) \geqslant 0$. Moreover $C^D(x) = 0$ for $x \in \partial\Sigma \cup \Sigma^c$. Thus

$$\sigma(x) = \left(\Delta C_\rho^D\right)(x) = \sum_{\{j\,:\,|j-x|=1\}} C_\rho^D(j) \geqslant 0 \tag{B.8}$$

By the lattice version of Gauss' theorem,

$$\sum_{x\in\Sigma} \sigma(x) = \sum_{x\in\partial\Sigma} \sigma(x) = \sum_{x\in\Sigma} \rho(x) \tag{B.9}$$

Thus, combining (B.8) and (B.9) and using *linearity*, i.e.,

$$\sigma_{\rho_1+\cdots+\rho_n} = \sigma_{\rho_1} + \cdots + \sigma_{\rho_n} \tag{B.10}$$

we find

$$\begin{aligned}\sum_{x\in\partial\Sigma} |\sigma(x)| &\leqslant \sum_{x\in\partial\Sigma}\sum_{y\in\Sigma} |\sigma_{\rho\cdot\delta_y}(x)| \\ &= \sum_{y\in\Sigma} |\rho(y)|\end{aligned} \tag{B.11}$$

Next, we compute *electrostatic energies*. We set

$$\begin{aligned} E_\rho &= \tfrac{1}{2}\sum_{x,y}\rho(x)C(x-y)\rho(y) \\ &= \tfrac{1}{2}\sum_{x}\rho(x)C_\rho(x) \\ &= \tfrac{1}{2}\sum_{x}(-\Delta C_\rho)(x)C_\rho(x) \\ &= \tfrac{1}{2}\sum_{x}(\nabla C_\rho)^2(x) \end{aligned} \tag{B.12}$$

For the purpose of renormalizing the activity of small dipoles we wish to compute $E_\rho - E_\sigma$. Using (B.12) and summation by parts we find

$$\begin{aligned} E_\rho - E_\sigma &= \tfrac{1}{2}\Sigma\nabla(C_\rho + C_\sigma)(x)\nabla(C_\rho - C_\sigma)(x) \\ &= \tfrac{1}{2}\Sigma[\rho(x)+\sigma(x)]C_{\rho-\sigma}(x) \\ &= \tfrac{1}{2}\Sigma\rho(x)C_{\rho-\sigma}(x) \end{aligned} \tag{B.13}$$

and we have used (B.1) and (B.3). Thus

$$\begin{aligned} E_\rho - E_\sigma &\geqslant E_\rho - \tfrac{1}{2}\sum_{y}\Big|\sum_{x}\rho(x)C(x-y)\Big||\sigma(y)| \\ &\geqslant E_\rho - \max_{y\in\partial\Sigma}\tfrac{1}{2}|\Sigma\rho(x)C(x-y)|\Big(\sum_{z\in\Sigma}|\rho(z)|\Big) \end{aligned} \tag{B.14}$$

which follows from (B.11).

We now show how these considerations can be applied to renormalize the fugacity of isolated dipoles. We note that, for arbitrary ρ,

$$\langle e^{i\phi(\rho)}\rangle_{\beta C} = e^{-\beta E_\rho}$$

so that

$$\frac{\langle e^{i\phi(\rho)}\rangle_{\beta C}}{\langle e^{i\phi(\sigma)}\rangle_{\beta C}} \leqslant e^{-\beta E_\rho}\exp\left\{\frac{1}{2}\max_{y\in\partial\Sigma}\Big|\sum_{x}\rho(x)C(x-y)\Big|\Big[\sum_{z\in\Sigma}|\rho(z)|\Big]\right\} \tag{B.15}$$

Now let ρ be the charge density of an *isolated* dipole moment r located at the origin of $\mathbb{Z}^2$, i.e.,

$$\rho(y) = \delta(y) - \delta(y-r) \tag{B.16}$$

Choose for Σ, e.g., a sphereical region centered at the point $r/2$ of mean radius $|r|$. Then

(i) $E_\rho = -C(0,r) \approx (1/2\pi)\log|r|$
(ii) $\Sigma|\rho(x)| = 2$
(iii) $\max_{y\in\partial\Sigma}|\Sigma\rho(x)C(x-y)| \leqslant K$

for some constant K independent of $|r|$. Thus

$$\exp \beta\left[E_{\rho}-E_{\sigma}\right]=\frac{\langle e^{i\phi(\rho)}\rangle_{\beta C}}{\langle e^{i\phi(\sigma)}\rangle_{\beta C}} \leqslant \exp\left[-(\beta/2\pi)\log|r|+\beta K\right] \quad \text{(B.17)}$$

By construction, see (B.2), $C_{\rho}(x)=C_{\sigma}(x)$, for $x \in \partial\Sigma \cup \Sigma^{c}$. Moreover, the dipole is *isolated*, in the sense that there is no other dipole inside Σ. Therefore we may replace

$$1+z\cos(\delta_{r}\phi)(0) \equiv 1+z\cos\phi(\rho)$$

by

$$\begin{gathered} 1+\bar{z}\cos\phi(\sigma), \text{ with } \sigma \text{ supported on } \partial\Sigma, \\ \sum_{x\in\partial\Sigma}\sigma(x)=0, \qquad \sum_{x\in\partial\Sigma}|\sigma(x)| \leqslant 2, \\ \text{and } \bar{z} \leqslant z\exp\left[-(\beta/2\pi)\log|r|+\beta K\right] \end{gathered} \quad \text{(B.18)}$$

We have found a *purely electrostatic* substitute for the technique used to prove Lemmas 5.3 and 5.4.

NOTE ADDED IN PROOF

The authors have rigorously established a phase transition for the two-dimensional Coulomb gas and plane rotator along the lines indicated in Section 5. Details will appear elsewhere.

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